Choquet simplices as the set of invariant probability measures of a postcritical set.

María Isabel Cortez

Departamento de Matemática y Ciencia de la Computación
Universidad de Santiago.

July 6, 2009.

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- $\mathcal{M}_T(X)$ is a metrizable Choquet simplex.
- Every metrizable Choquet simplex can be realized as $\mathcal{M}_T(X)$ for some Toeplitz subshift (X, T) (*T. Downarowicz, 91*).

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The result is true if we replace the family of logistic maps for any other full family.

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- $$P(t\log|f'|) = \sup_{\mu\in\mathcal{M}_f([0,1])}\{h_\mu - t\int \log|f'|d\mu\}.$$
- K is affine homeomorphic to the set of equilibrium states of $t\log|f'|$, with $t \geq 1$.

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- $T_Q :< n >\to< n+1 >$.
- If $Q(k) \to \infty$, then T_Q extends to a unique continuous map on Ω_Q (*Bruin, Keller, St. Pierre, 97*).

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To get $\Omega_{T_Q}(\Omega_Q) \approxeq K$, we use a **Bratteli-Vershik** representation of (Ω_Q, T_Q).

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Orbit equivalence $\Leftrightarrow$ There exists $F : X_1 \to X_2$ homeomorphism that induces an affine homeomorphism $F_* : \mathcal{M}_{T_1}(X_1) \to \mathcal{M}_{T_2}(X_2)$. (*Giordano, Putnam, Skau*).

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Do the minimal Cantor systems given by the restriction of a unimodal map to the omega-limit set associated to the postcritical set of the critical point realize all the class of orbital equivalence? What about the dimension groups associated to such systems?.

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- $q = (q_r)_{r\geq 0}$ is a sequence of increasing integers such that $q_0 = 0$ and

$$q_{r_n} - q_{r_n - 1} = a_{n,0} + \cdots + a_{n,n}.$$

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- $J_n = \{q_{r_n} + 1, \cdots, q_{r_{n+1} - 1}\}$.
- For each $0 \leq m \leq n$,

$$I_{n,m} = \{q_{r_n - 1} + 1 + \sum_{i=0}^{m-1} a_{n,i}, \cdots, q_{r_n - 1} + \sum_{i=0}^{m} a_{n,i}\},$$

and

$$J_{n,m} = \{q_{r_n + m} + 1, \cdots, q_{r_n + m + 1}\}.$$

Definition of Q.

$$Q = Q_{(\sigma,a,q)} = \sum_{n\in\mathbb{N}} \sum_{m=0}^{n} q_{r_{n-1}+m}(\mathbf{1}_{I_{n,m}} + \mathbf{1}_{J_{n,\sigma_n^{-1}(m)}}).$$

Theorem A.

If

$$\prod_{\substack{r \geq 1 \\ r \notin \{r_n : n \in \mathbb{N}\}}} \left(1 - \frac{S_{q_{r-1}}}{S_{q_r}}\right) > 0$$

then $\mathcal{M}_{T_Q}(\Omega_Q)$ and $\mathcal{M}_{f|_{\omega(c)}}(\omega(c))$ are affine homeomorphic to

$$\varprojlim(\triangle_n, A_n) = \triangle_0 \xleftarrow{A_0} \triangle_1 \xleftarrow{A_1} \triangle_2 \xleftarrow{A_2} \cdots,$$

where $\triangle_n = \{(x_0, \cdots, x_n) \in \mathbb{R}_+^{n+1} : \sum_{i=0}^n x_i = 1\}$ and $A_n = [\vec{\alpha}_n | \vec{e}_{\sigma_n(0)} | \cdots | \vec{e}_{\sigma_n(n)}]$.

Proof of the Main Theorem

We use Theorem A, the precedent Lemma and the next theorem.

Theorem

(Lazar, Lindenstrauss, 1971) Let K be an infinite dimensional metrizable Choquet simplex. Then there exists a sequence of surjective linear maps $(A_n)_{n\geq 0}$ such that $A_n : \triangle_{n+1} \to \triangle_n$ and such that K is affine homeomorphic to $\lim_{\leftarrow n}(\triangle_n, A_n)$.