Invariant measures for non minimal tiling systems.

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- There exists a finite set $\mathcal{A}$ of tiles, such that every tile in $\mathcal{T}$ is equal to some tile in $\mathcal{A}$, up to translations. The elements of $\mathcal{A}$ are called (**prototiles**). If A_1 and A_2 are equal up to translation, we say they are **equivalent**.

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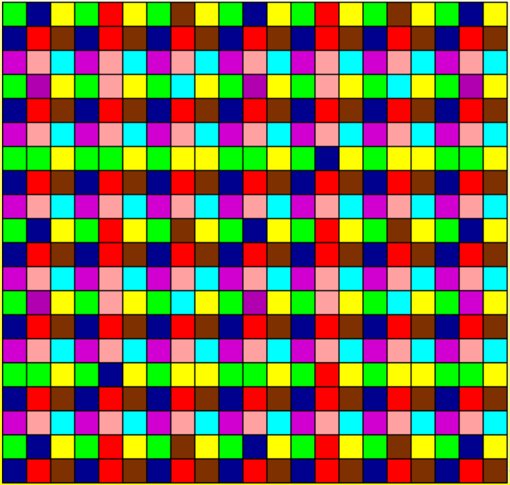
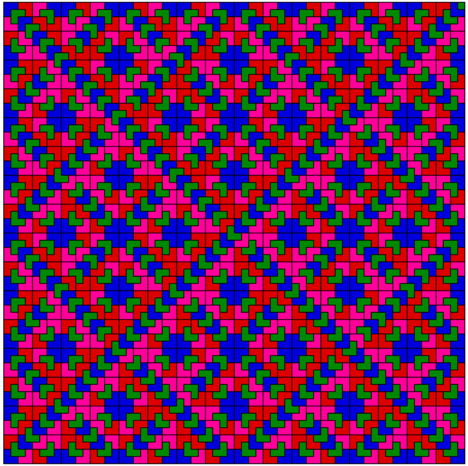
- A tile t could be a pair $(\mathrm{supp}(t), c_t)$, where $\mathrm{supp}(t)$ is a closed set as before, and c_i is a label or color. In this case, A_1 and A_2 are equivalent if their colors are equal and their supports are equivalent.

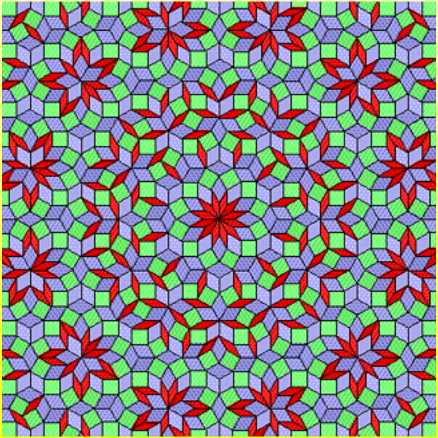
Tiling.



Chair tiling



Penrose tiling.



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- The set $X_{\mathcal{A}}$ could be empty. We assume that $X_{\mathcal{A}} \neq \emptyset$.
- $\mathbb{R}^d$ acts on $X_{\mathcal{A}}$ by translations:

$$\begin{array}{rcl} \mathbb{R}^d \times X_{\mathcal{A}} & \rightarrow & X_{\mathcal{A}} \\ (\vec{v}, \mathcal{T}) & \rightarrow & \mathcal{T} - \vec{v}, \end{array}$$

where $\mathcal{T} = \{t_n : n \geq 0\}$ and $\mathcal{T} - \vec{v} = \{t_n - \vec{v} : n \geq 0\}$.

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- For every $\vec{v} \in \mathbb{R}^d$, the translation $\mathcal{T} \to \mathcal{T} - \vec{v}$ is a homeomorphism (with respect to the topology generated by the metric d).

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A **tiling system** is a closed and translation-invariant set $X \subseteq X_{\mathcal{A}}$, equipped with the restriction of the $\mathbb{R}^d$-action to X. For instance, $\Omega_{\mathcal{T}} = \overline{\{\mathcal{T} - \vec{v} : \vec{v} \in \mathbb{R}^d\}}$.

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- X is compact if and only if X verifies FPC.

Finite pattern condition.

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- Let $\mathcal{A} = \{\square, \blacksquare\}$. We define

$$X = \{\mathcal{T} \in X_{\mathcal{A}} : \text{ the tiles in } \mathcal{T} \text{ meet side to side } \}$$

X is a tiling system verifying FPC.

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A function $\omega : \mathcal{A} \rightarrow \mathcal{A}^*$ is a **substitution** if there exists an expansive linear map $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}^d$ such that for every $A \in \mathcal{A}$,

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We have $\omega = S \circ \varphi$, where S is a subdivision rule.

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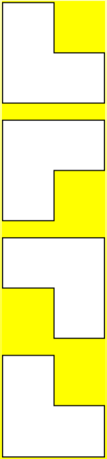

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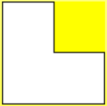

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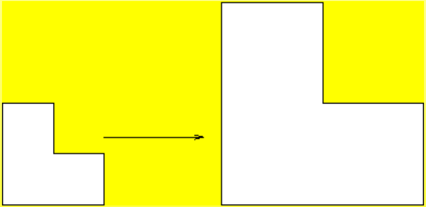

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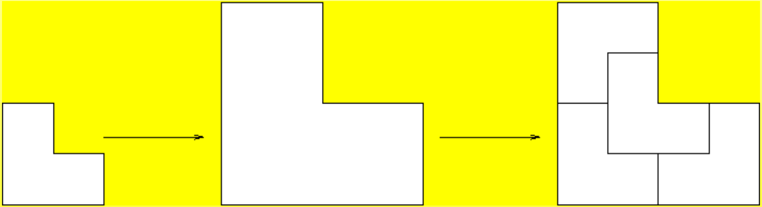

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In the same way, ω is extended to $X_{\mathcal{A}}$:

$$\omega(\mathcal{T}) = \{\omega(t_n) : n \in \mathbb{N}\} \text{ for } \mathcal{T} = \{t_n : n \in \mathbb{N}\} \in X_{\mathcal{A}}.$$

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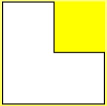

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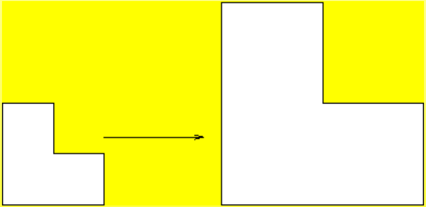

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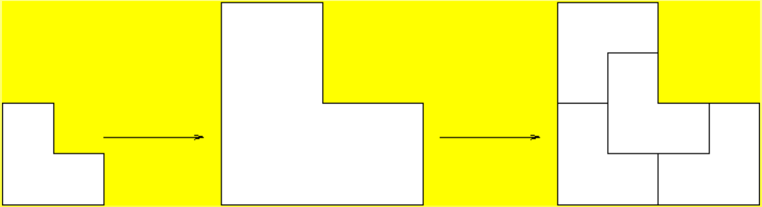

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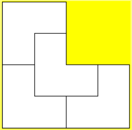

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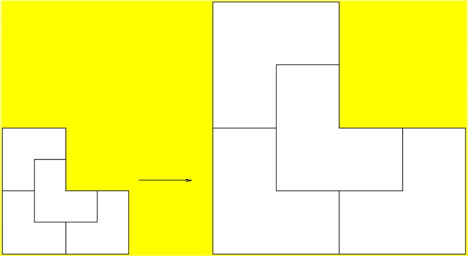

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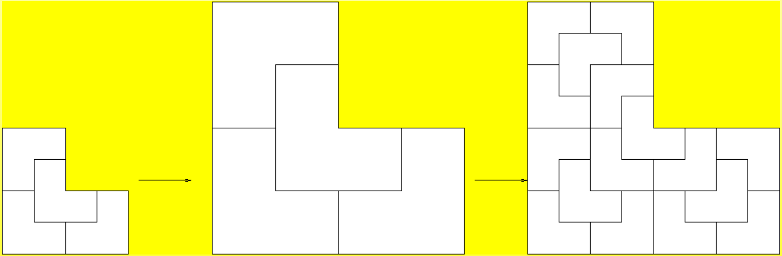

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The **substitution tiling space** associated to ω is the set

$$X_{\mathcal{A},\omega} = \{\mathcal{T} \in X_{\mathcal{A}} : \forall \text{ patch } P \subseteq \mathcal{T},\ \text{there exist } A \in \mathcal{A}, n > 0, \vec{x} \in \mathbb{R}^d \\ \text{such that } P + \vec{x} \in \omega^n(A)\}$$

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We assume that $X_{\mathcal{A},\omega}$ verifies FPC (compact).

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The associated matrix

$$M_\omega = \begin{pmatrix} 9 & 1 \\ 0 & 8 \end{pmatrix}.$$

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Question: How can we describe the sigma-finite invariant measures of $(X_{\mathcal{A},\omega}, \mathbb{R}^d)$ in the non primitive case?

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Let $A_1, \cdots, \mathcal{A}_l$ be the equivalences classes of $\sim$ (if ω is primitive then $\mathcal{A}$ is the only equivalence class).

Substitution matrix.

After to arrange the elemnts of $\mathcal{A}$ and to take powers of ω, the matrix M_ω looks like

$$M_\omega = \begin{pmatrix} M_1 & 0 & \cdots & 0 & X & X & X \\ 0 & \ddots & \ddots & \vdots & X & X & X \\ & \ddots & \ddots & 0 & X & X & X \\ \vdots & & \ddots & M_m & X & X & X \\ \vdots & & & \ddots & M_{m+1} & X & X \\ \vdots & & & & \ddots & \ddots & X \\ 0 & \cdots & & \cdots & \cdots & 0 & M_l \end{pmatrix}$$

(see *Introduction to symbolic dynamics and coding.* Lind and Marcus, 95)

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Theorem

(C. Solomyak) The finite invariant measures of $(X_{\mathcal{A},\omega}, \mathbb{R}^d)$ are supported on the minimal components. Thus if there are m minimal components, then $(X_{\mathcal{A},\omega}, \mathbb{R}^d)$ has exactly m ergodic probability measures.

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Proof. Ergodic Theorem and the Perron-Frobenius Theorem for matrices.

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With the induced topology, Γ is totally disconnected with base $\{C_P\}_P$.

Sigma-finite invariant measures

A **transverse measure** of $X_{\mathcal{A},\omega}$ is a Borel measure ν on Γ such that $\nu(U) = \nu(U - \vec{v})$, for every Borel set U and $\vec{v} \in \mathbb{R}^d$ such that $U - \vec{v} \subseteq \Gamma$.

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- There is a one-to-one correspondence between sigma-finite invariant measures and sigma-finite transversal measures.
- We call μ^T the transversal measure associated to the invariant measure μ. We have

$$\mu(C_P + \Theta) = \mu^T(C_P)\mathrm{vol}(\Theta),$$

  for every sufficiently small open set $\Theta \subseteq \mathbb{R}^d$.

Sigma-finite invariant measures

Under some assumptions on ω:

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Theorem

(C., Solomyak) For every $m+1 \leq i \leq l$ (such that M_i is primitive) there exists a sigma-finite ergodic measure μ_i supported on

$$Y_i = \{\mathcal{T} \in X_{\mathcal{A},\omega} : \mathcal{T} \text{ has a tile equivalent to some prototile in } \mathcal{A}_i\},$$

such that $0 < \mu_i^T(C_A) < \infty$, for every $A \in \mathcal{A}_i$. Conversely, if μ is a sigma-finite ergodic invariant measure such that $0 < \mu^T(C_A) < \infty$ for some $A \in \mathcal{A} \setminus \bigcup_{j=1}^m \mathcal{A}_j$, then μ is equal to some μ_i, up to multiplication by a constant.

Proof.

We assume that ω has *non-periodic border*:

$$\forall \mathcal{A} \in \mathcal{A}_{nonp},\ \partial(support(\omega(A))) \subseteq support(\omega(A)|_{\mathrm{nonp}}),$$

where $\mathcal{A}_{per} \subseteq \mathcal{A}$ are the prototiles which appears in periodic tilings belonging to minimal componentes, and $\mathcal{A}_{nonp} = \mathcal{A} \setminus \mathcal{A}_{per}$.

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The previous condition implies **recognizability:**

$\mathcal{T} \in X_{\mathcal{A},\omega}$ with a tile equivalent to a $A \in \mathcal{A}_{nonp} \Rightarrow |\omega^{-1}\{\mathcal{T}\}| = 1$.

Proof.

Recognizability implies that for every invariant measure μ

$$(\mu^T(C_A))_{A\in\mathcal{A}'} \in core(M_\omega) = \bigcap_{n\geq 0} M_\omega^n|_{\mathcal{A}'}((\overline{\mathbb{R}}^d)^+),$$

where $\mathcal{A}' = \bigcup_{j=m+1}^{l} \mathcal{A}_j$.

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The extra condition *for every* $m+1 \leq i \leq l$ *there exist* $A \in \mathcal{A}_i$ *and* $n > 0$ *such that the interior of* $\omega^n(A)$ *has a tile equivalent to* A implies every non trivial generator of $core(M_\omega)$ defines a sigma-finite ergodic transverse measure as we want.