Realization of some orbit equivalence classes.

María Isabel Cortez

Departamento de Matemática y Ciencia de la Computación
Universidad de Santiago de Chile.

August, 2011.

Framework

We deal with dynamical systems (X, T) such that:

- X is a Cantor set,
- $T : X \to X$ is a homeomorphism (sometimes one-to-one up to a negligible set),
- (X, T) is minimal (every orbit $o_T(x) = \{T^n(x) : n \in \mathbb{Z}\}$ is dense).

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We call (X, T) a **minimal Cantor system**.

Examples

- Substitution subshifts asociated to primitive substitutions (expansive, uniquely ergodic, zero entropy).
- Odometers (equicontinuous, uniquely ergodic, zero entropy).
- Toeplitz subshifts (expansive, any possible set of invariant measures, any possible entropy).

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Orbit equivalence

Two minimal Cantor systems (X_1, T_1) and (X_2, T_2) are (topologically) **orbit equivalent** if there exists a homeomorphism $F : X_1 \to X_2$ such that

- Orbit equivalence is an equivalence relation on the set of minimal Cantor systems,
- We can regroup minimal Cantor systems in orbit equivalence classes.

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General problem: to find a particular family of minimal Cantor systems having a representative element from each orbit equivalence class.

Invariants associated to the orbit equivalence.

The set of **invariant probability measures** $\mathcal{M}(X, T)$ of (X, T).

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Remark: If $\mathcal{F}$ is a family having a representative element from each orbit equivalence class, then for every metrizable Choquet simplex K there exist $(X, T) \in \mathcal{F}$ and an affine bijection between $\mathcal{M}(X, T)$ and K (the converse is not true!!).

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The **ordered group with unit** associated to (X, T):

$$\mathcal{G}(X,T) = (G(X,T), G(X,T)^{+}, u(X,T)),$$

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$\mathcal{G}(X,T)$ is a simple dimension group (see [Herman, Putnam, Skau 92]).

Suppose that (X, T) is uniquely ergodic.

- The homomorphism of groups

$$\begin{aligned} \phi : C(X,\mathbb{Z}) &\longrightarrow \mathbb{R} \\ f &\longrightarrow \int f d\mu \end{aligned}$$

  induces an isomorphism of ordered group with unit between $\mathcal{G}(X, T)$ and $(\phi(C(X,\mathbb{Z})), \phi(C(X,\mathbb{Z})) \cap \mathbb{R}^+, 1)$.
- Conversely, if $\mathbb{Z} \subseteq \Gamma \subseteq \mathbb{R}$ is a countable subgroup, then there exists (X, T) such that $\mathcal{G}(X, T)$ is isomorphic to $(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$ (consequence of [Herman, Putnam, Skau 92]).

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Remark: a family of minimal Cantor systems $\mathcal{F}$ has a representative element from each uniquely ergodic orbit equivalence class iff $\mathcal{F}$ realizes every ordered group of the kind $(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$.

Comments

- The ordered group associated to a Toeplitz subshift has a non trivial rational part. Thus the family of Toeplitz subshifts does not realizes every orbit equivalence class (for example, they can not be orbit equivalent to a Sturmian subshift).
- From [Putnam, Schmidt, Skau 86] and [Giordano, Putnam, Skau 95]: Denjoy systems realize every uniquely ergodic orbit equivalence class (and these are the unique orbit equivalence classes that they realize).
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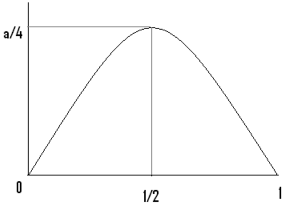
Generalized odometer associated to a unimodal map f.

Let $f : [0,1] \to [0,1]$ be the **unimodal map** given by $f(x) = ax(1-x)$, where $a \in (0,4]$.



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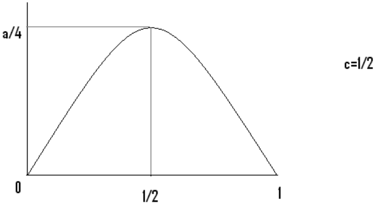

The orbit of c defines an increasing sequence of integers $(S_k)_{k \geq 0}$ (called "cutting times") verifying: $S_0 = 1$, $S_{k+1} > S_k$, and $S_{k+1} \leq 2S_k$.

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Before to define the generalized odometer associated to f:

- Let $k \geq 0$ be the unique integer satisfying $S_k \leq n < S_{k+1}$.
- We set $\langle n \rangle_k = 1$ and $m = n - S_k$. If $m = 0$ we stop. If $m > 0$ we go the precedent step with $n = m$.
- We get $n = n = \sum_{i \geq 0} \langle n \rangle_i S_i$, defining $\langle n \rangle_i = 0$ for every $i \geq 0$ such that $\langle n \rangle_i$ was not defined before.

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From [Bruin, Keller, St.Pierre 97]:

- $T_f : \{\langle n \rangle : n \in \mathbb{N}\} \to \{\langle n \rangle : n \in \mathbb{N}\}$ given by $T(\langle n \rangle) = \langle n+1 \rangle$, extends to a unique continuous map on

$$\Omega_f = \overline{\{\langle n \rangle : n \in \mathbb{N}\}}.$$

- The system (Ω_f, T_f) is the **generalized odometer** associated to f.
- (Ω_f, T_f) is minimal and T_f^{-1} is well defined on $\Omega_f \setminus \langle 0 \rangle$.
- There exists a topological factor $\pi : (\Omega_f, T_f) \to (\omega(c), f|_{\omega(c)})$, where $\omega(c)$ is the omega limit set of c (and sometimes, this factor map is one-to-one on a full measure set).

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The Bratteli-Vershik representation of the generalized odometer [Bruin 03]:

- There exists $Q : \mathbb{N} \to \mathbb{N}$ such that $Q(0) = 0$, $Q(k) < k$ and $S_k = S_{k-1} + S_{Q(k)}$ (the "kneading map" associated to f).
- The phase space of the generalized odometer:

$$\Omega_f = \{(x_n)_{n\geq 0} \in \{0,1\}^{\mathbb{N}} : x_k = 1 \Rightarrow x_j = 0 \forall Q(k+1) \leq j \leq k-1\}.$$

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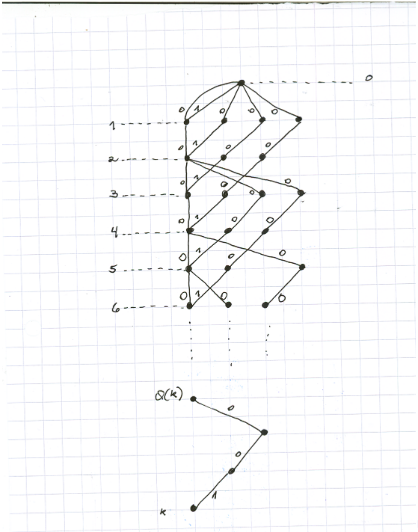

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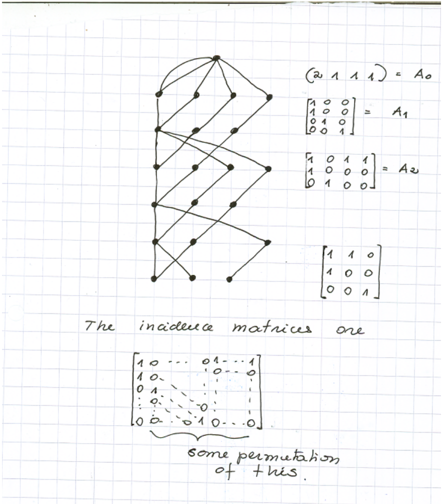

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Remark: Every usual odometer is a generalized odometer defined by a unimodal map.

Theorem

(C, Rivera-Letelier 2010) For every metrizable Choquet simplex K, there exists f such that there is an affine homeomorphism from K to $\mathcal{M}(\Omega_f, T_f)$ and $\mathcal{M}(\omega(c), f|_{\omega(c)})$.

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The family $\mathcal{F}$ of all the (natural extensions of) generalized odometers coming from unimodal maps is a candidate to realize every orbit equivalence class.

Uniquely ergodic orbit equivalence classes.

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(C, Rivera-Letelier 2011) For every uniquely ergodic minimal Cantor system (X, T) *there exists a unimodal map* f*, such that* (X, T) *is orbit equivalent to (the natural extension of)* (Ω_f, T_f).

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(C, Rivera-Letelier 2011) For every uniquely ergodic minimal Cantor system (X, T) *there exist a unimodal map* f *and a continuous orbit preserving map* $h : X \to \omega(c)$*, where* c *is the critical point of* f*, such that* h^{-1} *is defined outside the backward orbit of* c*.*

Uniquely ergodic orbit equivalence classes.

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(C, Rivera-Letelier 2011) For every uniquely ergodic minimal Cantor system (X, T) *there exists a unimodal map* f*, such that* (X, T) *is orbit equivalent to (the natural extension of)* (Ω_f, T_f)*.*

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(C, Rivera-Letelier 2011) For every uniquely ergodic minimal Cantor system (X, T) *there exist a unimodal map* f *and a continuous orbit preserving map* $h : X \to \omega(c)$*, where* c *is the critical point of* f*, such that* h^{-1} *is defined outside the backward orbit of* c*.*

Open question: realization of all the orbit equivalence classes.

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$$\begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} = \begin{pmatrix} a_1 & 1 & \cdots & 1 \\ \vdots & 0 & & \vdots \\ a_{d-1} & \vdots & & 1 \\ 1 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} x'_1 \\ \vdots \\ x'_{d'} \end{pmatrix},$$

up to permutations of the last $d-1$ columns, and up to some $d-d'$ columns of the last $d-1$ columns of the matrix.

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- Let define $y_d = x_d$ and $b_d = 1$.
- For $1 \leq j \leq d-1$ we define

$$y_j = x_d \left\{ \frac{x_j - x_{j+1}}{x_d} \right\}, b_j = \left[\frac{x_j - x_{j+1}}{x_d} \right] \text{ and } a_j = \sum_{k=j}^{d} b_k.$$

- After some permutations of the indices we get $y_1 = x_d > y_2 \geq \cdots \geq y_{d'} > 0, \cdots, 0$.
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The matrices A_n are a product of "admissible" matrices.

Ordered group with unit

To define an "admissible" Bratteli diagram (V, E) such that $K^0(V, E)/inf(K^0(V, E))$ is isomorphic to $(\Gamma, \Gamma \cap \mathbb{R}^+, 1)$:

- $\Gamma =< \{\alpha_l : l \geq 0\} >$, $\alpha_0 = 1$ and $\alpha_1 \in (0, 1/2) \setminus \mathbb{Q}$.
- $\Gamma_l =< \{\alpha_0, \cdots, \alpha_l\} >$, d_l the rank of Γ_l.
- $y^{(0)} = (1)$, $y^{(1)} = (1 - \alpha_1, \alpha_1)$ and $M_0 = (1 \;\; 1)$.
- $y^{(l)}$ is a strictly decreasing and positive vector whose coordinates are a base of Γ_l.
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The dimension group of the natural extension of the generalized odometer is the same.