Dimension groups and generalized odometers.

(Work in progress)

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Framework

We deal with dynamical systems (X, T, G) such that:

- X is a Cantor set,
- G is a countable amenable group,
- T is an action of G on X such that every $T^g : X \to X$, given by $T^g(x) = T(g, x)$, is a homeomorphism.
- The action is free ($T^g(x) = x$ implies $g = e_G$) and minimal (every orbit $o_T(x) = \{T^g(x) : g \in G\}$ is dense).

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We call (X, T, G) a **minimal Cantor system**.

Orbit equivalence

Two minimal Cantor systems (X_1, T_1, G_1) and (X_2, T_2, G_2) are (topological) **orbit equivalent** if there exists a homeomorphism $F : X_1 \to X_2$ such that

- Orbit equivalence is an equivalence relation on the set of minimal Cantor systems,
- We can regroup minimal Cantor systems in orbit equivalence classes.

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- We can regroup minimal Cantor systems in orbit equivalence classes.

General problem: to find a particular family of minimal Cantor systems having a representative element from each orbit equivalence class.

Invariants associated to orbit equivalence.

Let (X, T, G) be a minimal Cantor system.

- $H = C(X, \mathbb{Z})/\{f \in C(X, \mathbb{Z}) : \int f d\mu = 0, \forall \mu \in \mathcal{M}(X, T, G)\}$,
- $H^+ = \{[f] : f \geq 0\}$,
- $u = [1_X]$.

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Important results about the invariants.

From [Giordano, Matui, Putnam, Skau, 2010] and [Giordano, Putnam, Skau 95]:

- Every $\mathbb{Z}^d$-minimal Cantor system is orbit equivalent to a $\mathbb{Z}$-minimal Cantor system.
- $(X_1, T_1, \mathbb{Z}^d)$ and $(X_2, T_2, \mathbb{Z}^r)$ are orbit equivalent iff $\mathcal{G}(X_1, T_1, \mathbb{Z}^d)$ and $\mathcal{G}(X_2, T_2, \mathbb{Z}^r)$ are isomorphic as ordered groups with unit.
- $(X_1, T_1, \mathbb{Z}^d)$ and $(X_2, T_2, \mathbb{Z}^r)$ are orbit equivalent iff there exists a homeomorphism $F : X_1 \to X_2$ that induces a bijection between $\mathcal{M}(X_1, T_1, \mathbb{Z}^d)$ and $\mathcal{M}(X_2, T_2, \mathbb{Z}^r)$.

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Unknown: these results for any amenable group action.

Important results about the invariants.

From [Herman, Putnam, Skau 92] and [Giordano, Putnam, Skau 95]:

- The ordered abelian group with unit associated to a $\mathbb{Z}$-minimal Cantor system is a simple acyclic dimension group (with trivial infinitesimal).
- For every simple acyclic dimension group $\mathcal{G}$ (with trivial infinitesimal) there exists a $\mathbb{Z}$-minimal Cantor system whose ordered abelian group with unit is isomorphic to $\mathcal{G}$.

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Unknown: the range for $\mathbb{Z}^d$-actions, for $d > 1$.

Let $\mathcal{C}$ be a family of minimal Cantor systems.

- If every Choquet simplex can be seen as the set of invariant probability measures of a system in $\mathcal{C}$, then $\mathcal{C}$ is a good candidate to have a representative element from each orbit equivalence class.
- Suppose that $\mathcal{C}$ is a candidate, then (at least for $\mathbb{Z}^d$-actions) it is enough to show that for every simple acyclic dimension group $\mathcal{G}$, there exists a system in $\mathcal{C}$ with an associated ordered group with unit isomorphic to $\mathcal{G}$.

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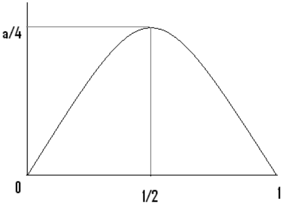
Generalized odometer associated to a unimodal map f.

Let $a \in (0, 4]$ and f_a be the **unimodal map** given by $f_a(x) = ax(1 - x)$ for $x \in [0, 1]$.



c=1/2

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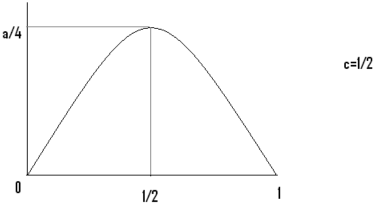

f_a has a **sequence of cutting times** $(S_k)_{k\geq 0}$ verifying: $S_0 = 1$, $S_{k+1} > S_k$ integers, and $\frac{S_{k+1}}{S_k} \leq 2$.

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For every $n \in \mathbb{N}$, there exists $< n >= (< n >_i)_{i \geq 0} \in \{0, 1\}^{\mathbb{N}}$ such that $n = \sum_{i \geq 0} < n >_i S_i$.

- Let $k \geq 0$ be the unique integer satisfying $S_k \leq n < S_{k+1}$.
- We set $< n >_k = 1$ and $m = n - S_k$. If $m = 0$ we stop. If $m > 0$ we go the precedent step with $n = m$.
- We get $n = n = \sum_{i \geq 0} < n >_i S_i$, defining $< n >_i = 0$ for every $i \geq 0$ such that $< n >_i$ was not defined before.

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From [Bruin, Keller, St.Pierre 97]:

- $T_a : \{< n >: n \in \mathbb{N}\} \to \{< n >: n \in \mathbb{N}\}$ given by $T_a(< n >) =< n+1 >$, extends to a unique continuous map on
$$\Omega_a = \overline{\{< n >: n \in \mathbb{N}\}}.$$
- The system (Ω_a, T_a) is the **generalized odometer** associated to f_a.
- (Ω_a, T_a) is minimal.
- There exists a topological factor $\pi : (\Omega_a, T_a) \to (\omega(c), f_a|_{\omega(c)})$, where $\omega(c)$ is the omega limit set of $c = 1/2$.

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Theorem

(C., Rivera-Letelier 2010) For every metrizable Choquet simplex K, there exists $a \in (0, 4]$ such that there is an affine bijection from K to $\mathcal{M}(\Omega_a, T_a)$ and $\mathcal{M}(\omega(c), f_a|_{\omega(c)})$.

- The family $\mathcal{C}$ of all the generalized odometers coming from unimodal maps is a candidate to realize every orbit equivalence class.
- (Downarowicz 1991) the family of Toeplitz flows realizes every Choquet simplex as the set of invariant measures. Nevertheless the associated dimension groups have always non trivial rational subgroups.

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There exists a countable acyclic subgroup H of $\mathbb{R}$ such that $\mathcal{G}(X, T)$ is isomorphic to $(H, H \cap \mathbb{R}^+, 1)$.

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Question: Let H be a countable acyclic subgroup of $\mathbb{R}$. Does it exist $a \in (0, 1]$ such that $\mathcal{G}(\Omega_a, T_a)$ is isomorphic to $(H, H \cap \mathbb{R}^+, 1)$?

Partial answer.

Theorem

(C., Rivera-Letelier) Let Γ be a subgroup of $\mathbb{Q}$ and let $(\alpha_i)_{i\in I}$ be a finite or countable collection of rationally independent positive numbers. If

$$H = \bigoplus_{i\in I} \Gamma\alpha_i,$$

then there exists $a \in (0,4]$ such that the dimension group associated to (Ω_a, T_a), is isomorphic to

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- What about the rest of the groups H?.
- What about the rest the minimal Cantor systems?

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then there exists $a \in (0,4]$ such that the dimension group associated to (Ω_a, T_a), is isomorphic to

$$(H, H\cap\mathbb{R}^+, 1).$$

- What about the rest of the groups H?.
- What about the rest the minimal Cantor systems?

Some corollaries.

- For every minimal Cantor system $(X, T, \mathbb{Z}^d)$ such that $\mathcal{G}(X, T, \mathbb{Z}^d)$ is as in the theorem, there exists $a \in (0, 4]$ such that (the natural extension of) (Ω_a, T_a) is orbit equivalent to $(X, T, \mathbb{Z}^d)$.
- For every uniquely ergodic minimal Cantor system $(X, T, \mathbb{Z}^d)$ whose ordered group is divisible, there exists $a \in (0, 4]$ such that (the natural extension of) (Ω_a, T_a) is orbit equivalent to $(X, T, \mathbb{Z}^d)$.
- For every uniquely ergodic minimal Cantor system $(X, T, \mathbb{Z}^d)$ whose ordered group is free, there exists $a \in (0, 4]$ such that (the natural extension of) (Ω_a, T_a) is orbit equivalent to $(X, T, \mathbb{Z}^d)$.

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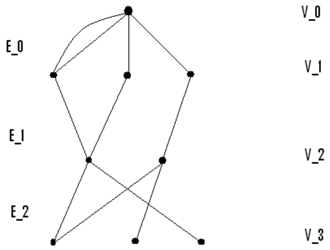
Proof ideas.

Principal tool: Bratteli-Vershik representation of minimal Cantor systems.



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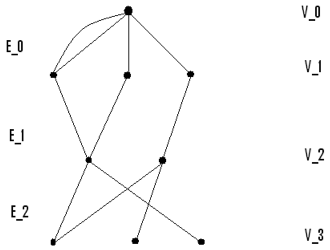

Bratelli diagram $B = (V, E)$, where $V = (V_n)_{n \geq 0}$ is the set of vertices and $E = (E_n)_{n \geq 0}$ is the set of edges.

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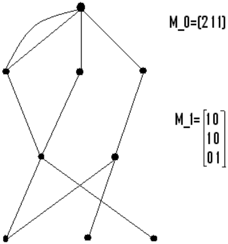

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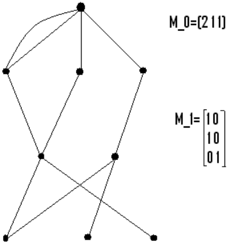

Sequence of incidence matrices $(M_n)_{n \geq 0}$, where $M_n(i, j)$ is the number of edges from i to j.

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The associated dimension group is isomorphic to

$$H = \lim_{n\rightarrow}(\mathbb{Z}^{|V_n|}, M_n^T) = \mathbb{Z} \xrightarrow{M_0^T} \mathbb{Z}^{|V_1|} \xrightarrow{M_1^T} \mathbb{Z}^{|V_2|} \xrightarrow{M_2^T} \cdots,$$

modulo the infinitesimal subgroup.

Proof ideas.

Given a sequence $(\alpha_i)_{i\in I}$ of rationally independent positive numbers, to find a Bratteli diagram $B = (V, E)$ verifying the following conditions:

- The dimension group

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  is isomorphic to $(\bigoplus_{i\in I}\mathbb{Z}\alpha_i, \mathbb{R}^+ \cap \bigoplus_{i\in I}\mathbb{Z}\alpha_i, 1)$.
- $B = (V, E)$ represents a generalized odometer.

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In order to get $(\bigoplus_{i\in I}\Gamma\alpha_i, \mathbb{R}^+\cap\bigoplus_{i\in I}\Gamma\alpha_i, 1)$, we place diagonal matrices among $(M_n)_{n\geq 0}$.

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$(X, \sigma|_X, G)$ is a **Toeplitz subshift** if $X = \overline{o_\sigma(x)}$, for some Toeplitz element $x \in \Sigma^G$.

Some results.

Theorem

(C., Petite) Let G be a residually finite, finitely generated amenable group G. For every Choquet simplex K there exists a Toeplitz subshift $(X, \sigma|_X, G)$ such that $\mathcal{M}(X, \sigma|_X, G)$ is affine homeomorphic to K.

- Given a $\mathbb{Z}$-Toeplitz subshift. Does it exist a G-Toeplitz subshift having an isomorphic associated ordered Abelian group with unit? (work in progress).
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Proof ideas.

Principal tool: Nested clopen partitions of Toeplitz subshifts.

- Let $(\Gamma_n)_{n\geq 0}$ be a decreasing sequence of normal finite index subgroups of G verifying $\cap_{n\geq 0}\Gamma_n = \{e\}$.
- Let $(F_n)_{n\geq 0}$ be a Følner sequence of fundamental domains such that $F_n \subset F_{n+1}$, $\bigcup_{n\geq 0} F_n = G$ and $F_{n+1} = \bigcup_{v\in F_{n+1}\cap\Gamma_n} vF_n$.
- Let $(M_n)_{n\geq 0}$ be a "suitable" sequence of positive integer matrices.
- For every $n \geq 0$, we construct "suitable" elements $B_{n,1}, \cdots, B_{n,k_n} \in \Sigma^{F_n}$.
- $\bigcap_{n\geq 0}[B_{n,1}]$ has only one element x, x is Toeplitz, and the set of invariant probability measures of $(\overline{o_\sigma(x)}, \sigma, G)$ is the inverse limit $\lim_{\leftarrow n}(\triangle_n, M_n)$.

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