SET OF INVARIANT MEASURES OF GENERALIZED TOEPLITZ SUBSHIFTS.

MARÍA ISABEL CORTEZ, SAMUEL PETITE

Abstract. We show that for every metrizable Choquet simplex K and for every group G, which is amenable, finitely generated and residually finite, there exists a Toeplitz G-subshift whose set of shift-invariant probability measures is affine homeomorphic to K. Furthermore, we get that for every integer $d \geq 1$ and every minimal Cantor system (X,T) whose dimension group is divisible, there exists a minimal Toeplitz $\mathbb{Z}^d$-subshift which is topologically orbit equivalent to (X,T).

1. Introduction

The *Toeplitz subshifts* are a rich class of symbolic systems introduced by Jacobs and Keane in [19], in the context of $\mathbb{Z}$-actions. Since then, they have been extensively studied and used to provide series of examples with interesting dynamical properties (see for example [6, 7, 16, 23]). Generalizations of Toeplitz subshifts and some of their properties to more general group actions can be found in [2, 4, 8, 20]. For instance, in [4] Toeplitz subshifts are characterized as the minimal symbolic almost 1-1 extensions of odometers (see [12] for this result in the context of $\mathbb{Z}$-actions). In this paper, we give an explicit construction that generalizes the result of Downarowicz in [6], to Toeplitz subshifts given by actions of groups which are amenable, residually finite and finitely generated. The following is our main result.

Theorem A. *Let G be an amenable, residually finite and finitely generated group. For every metrizable Choquet simplex K and any G-odometer O, there exists a Toeplitz G-subshift whose set of invariant probability measures is affine homeomorphic to K and such that it is an almost 1-1 extension of O.*

Typical examples of the groups G involved in this theorem are the finitely generated subgroups of upper triangular matrices in $GL(n,\mathbb{C})$.

The proof of Theorem A deals with combinatorics on Følner sequences and is independent on the known results for $G=\mathbb{Z}$ (see [6] or [16]).

Furethermore, we obtain some consequences in the orbit equivalence problem. Two minimal Cantor systems are (topologically) orbit equivalent, if there exists an orbit-preserving homeomorphism between their phase spaces. Giordano, Matui, Putnam and Skau show in [14] that every minimal $\mathbb{Z}^d$-action on the Cantor set is orbit equivalent to a minimal $\mathbb{Z}$-action. Such a result can not be extended to any countable group. For instance, by using the notion of cost, Gaboriau [13] proves that if two free actions of free groups $\mathbb{F}_n$ and $\mathbb{F}_p$ are (even measurably) orbit equivalent then their rank are the same i.e. $n=p$. It is still unknown for which groups the result in [14] is true and which are the $\mathbb{Z}$-orbit equivalence classes that the $\mathbb{Z}^d$-actions (or more general group actions) realize. We give a partial answer for the second question. As a consequence of the proof of Theorem A we obtain the following result.

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Theorem B. *Let (X, T) be a minimal Cantor system having an associated Bratteli diagram B which satisfies the equal path number property. For each $n \geq 1$, let V_n and r_n be the set of vertices of the level n of B and the number of edges arriving to a vertex v in V_n respectively. If there exists a constant $C > 0$ such that $Cr_n \geq |V_n|$ for each $n \geq 1$, then for every $d \geq 1$ there exists a Toeplitz $\mathbb{Z}^d$-subshift which is orbit equivalent to (X, T). In particular, every minimal Cantor system (X, T) whose dimension group is divisible, is orbit equivalent to a Toeplitz $\mathbb{Z}^d$-subshift.*

It remains open the problem if the previous result can be generalized to more general group actions and to any Toeplitz subshift given by a $\mathbb{Z}$-action.

This paper is organized as follows. Section 2 is devoted to introduce the basic definitions. In Section 3 we give a characterization of any Choquet simplex as an inverse limit defined by suitable sequences of matrices that we call "manageable". For an amenable discrete group G and a decreasing sequence of finite index subgroups of G with trivial intersection, we construct in Section 4 an associated sequence $(F_n)_{n\geq 0}$ of fundamental domains, so that it is Følner and each F_{n+1} is tilable by translated copies of F_n. Next we use the representation established in Section 3 and the previous Følner sequence to construct a Toeplitz G-subshift with a prescribed set of invariant probability measures. This construction improves and generalizes that one given in [3] for $\mathbb{Z}^d$-actions, and moreover, allows to characterize the associated dimension group. We use this result to prove Theorem B in the last section.

2. Basic definitions and background

In this article, by a *topological dynamical system* we mean a triple (X, T, G), where T is a continuous left action of a finitely generated group G on the compact metric space (X, d). For every $g \in G$, we denote T^g the homeomorphism that induces the action of g on X. The unit element of G will be called e. The system (X, T, G) or the action T is *minimal* if for every $x \in X$ the orbit $o_T(x) = \{T^g(x) : g \in G\}$ is dense in X. We say that (X, T, G) is a *minimal Cantor system* or a minimal Cantor G-system if (X, T, G) is a minimal topological dynamical system with X a Cantor set.
An *invariant probability measure* of the topological dynamical system (X, T, G) is a probability Borel measure μ such that $\mu(T^g(A)) = \mu(A)$, for every Borel set A. We denote by $\mathcal{M}(X, T, G)$ the space of invariant probability measures of (X, T, G).

2.1. **Subshifts.**

For every $g \in G$, denote $L_g : G \to G$ the left multiplication by $g \in G$. That is, $L_g(h) = gh$ for every $h \in G$. Let Σ be a finite alphabet. Σ^G denotes the set of all the functions $x : G \to \Sigma$. The (left) *shift action* σ of G on Σ^G is given by $\sigma^g(x) = x \circ L_{g^{-1}}$, for every $g \in G$. Thus $\sigma^g(x)(h) = x(g^{-1}h)$. We consider Σ endowed with the discrete topology and Σ^G with the product topology. Thus every σ^g is a homeomorphism of the Cantor set Σ^G. The topological dynamical system (Σ^G, σ, G) is called the full G-shift on Σ. For every finite subset D of G and $x \in \Sigma^G$, we denote $x|_D \in \Sigma^D$ the restriction of x to D. For $F \in \Sigma^D$ (F is a function from D to Σ) we denote by $[F]$ the set of all $x \in \Sigma^D$ such that $x|_D = F$. The set $[F]$ is called the *cylinder* defined by F, and it is a clopen set (both open and closed). The collection of all the sets $[F]$ is a base of the topology of Σ^G. A *subshift* or G-subshift of Σ^G is a closed subset X of Σ^G which is invariant by the shift action. The topological dynamical system $(X, \sigma|_X, G)$ is also called subshift or G-subshift. See [1] for details.

2.1.1. *Toeplitz G-subshifts.* An element $x \in \Sigma^G$ is a *Toeplitz sequence*, if for every $g \in G$ there exists a finite index subgroup Γ of G such that $\sigma^\gamma(x)(g) = x(\gamma^{-1}g) = x(g)$, for every $\gamma \in \Gamma$.

A subshift $X \subseteq \Sigma^G$ is a *Toeplitz subshift* or Toeplitz G-subshift if there exists a Toeplitz sequence $x \in \Sigma^G$ such that $X = \overline{o_\sigma(x)}$. Such subshifts are minimal because the Toeplitz sequences are regularly recurrent. See [4] and [20] for details about Toeplitz sequence and subshifts for G-actions.

2.2. **Inverse and direct limit.** Given a sequence of continuous maps $f_n \colon X_{n+1} \to X_n, n \geq 0$ on topological spaces X_n, we denote the associated *inverse limit* by

$$\begin{aligned} \lim_{\leftarrow n}(X_n, f_n) &= X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} X_2 \xleftarrow{f_2} \cdots \\ &:= \{(x_n)_n; x_n \in X_n,\ x_n = f_n(x_{n+1})\ \forall n \geq 0\}. \end{aligned}$$

Let us recall that this space is compact when all the spaces X_n are compact and the inverse limit spaces associated to any increasing subsequences $(n_i)_i$ of indices are homeomorphic.
In a similar way, we denote for a sequence of maps $g_n \colon X_n \to X_{n+1}, n \geq 0$ the associated *direct limit* by

$$\begin{aligned} \lim_{\rightarrow n}(X_n, g_n) &= X_0 \xrightarrow{g_0} X_1 \xrightarrow{g_1} X_2 \xrightarrow{g_2} \cdots \\ &:= \{(x, n), x \in X_n,\ n \geq 0\}/\sim, \end{aligned}$$

where two elements are equivalent $(x, n) \sim (y, m)$ if and only if there exists $k \geq m, n$ such that $g_k \circ \ldots \circ g_n(x) = g_k \circ \ldots \circ g_m(x)$. We denote by $[x, n]$ the equivalence class of (x, n). When the maps g_n are homomorphisms on groups X_n, then the direct limit inherits a group structure.

2.3. **Odometers.** A group G is said to be *residually finite* if there exists a nested sequence $(\Gamma_n)_{n\geq 0}$ of finite index normal subgroups such that $\bigcap_{n\geq 0} \Gamma_n$ is trivial. For every $n \geq 0$, there exists then a canonical projection $\pi_n \colon G/\Gamma_{n+1} \to G/\Gamma_n$. The *$G$-odometer* or *adding machine* O associated to the sequence $(\Gamma_n)_n$ is the inverse limit

$$O := \lim_{\leftarrow n}(G/\Gamma_n, \pi_n) = G/\Gamma_0 \xleftarrow{\pi_0} G/\Gamma_1 \xleftarrow{\pi_1} G/\Gamma_2 \xleftarrow{\pi_2} \cdots .$$

We refer to [4] for the basic properties of such a space. Let us recall that it inherits a group structure through the quotient groups G/Γ_n and it contains G as a subgroup thanks the injection $G \ni g \mapsto ([g]_n) \in O$, where $[g]_n$ denotes the class of g in G/Γ_n. Thus the group G acts by left multiplication on O. When there is no confusion, we call this action also odometer. It is equicontiuous, minimal and the left Haar measure is the unique invariant probability measure. Notice that this action is free: the stabilizer of any point is trivial. The Toeplitz G-subshifts are characterized as the subshifts that are minimal almost 1-1 extensions of G-odometers [4].

2.4. **Ordered groups.** An *ordered group* is a pair (H, H^+), such that H is a countable abelian group and H^+ is a subset of H verifying $(H^+) + (H^+) \subseteq H^+$, $(H^+) + (-H^+) = H$ and $(H^+) \cap (-H^+) = \{0\}$ (we use 0 as the unit of H when H is abelian). An ordered group (H, H^+) is a *dimension group* if for every $n \in \mathbb{Z}^+$ there exist $k_n \geq 1$ and a positive homomorphism $A_n : \mathbb{Z}^{k_n} \to \mathbb{Z}^{k_{n+1}}$, such that (H, H^+) is isomorphic to (J, J^+), where J is the direct limit

$$\lim_{\longrightarrow n}(\mathbb{Z}^{k_n}, A_n) = \mathbb{Z}^{k_0} \xrightarrow{A_0} \mathbb{Z}^{k_1} \xrightarrow{A_1} \mathbb{Z}^{k_2} \xrightarrow{A_2} \cdots ,$$

and $J^+ = \{[v, n] : a \in (\mathbb{Z}^+)^{k_n}, n \in \mathbb{Z}^+\}$. The dimension group is *simple* if the matrices A_n can be chosen strictly positive.
An *order unit* in the ordered group (H, H^+) is an element $u \in H^+$ such that for every $g \in H$ there exists $n \in \mathbb{Z}^+$ such that $g \leq nu$. If (H, H^+) is a simple dimension group then each

element in $H^+ \setminus \{0\}$ is an order unit. A *unital ordered group* is a triple (H, H^+, u) such that (H, H^+) is an ordered group and u is an order unit. An isomorphism between two unital ordered groups (H, H^+, u) and (J, J^+, v) is an isomorphism $\phi : H \to J$ such that $\phi(H^+) = J^+$ and $\phi(u) = v$. A *state* of the unital ordered group (H, H^+, u) is a homomorphism $\phi : H \to \mathbb{R}$ so that $\phi(u) = 1$ and $\phi(H^+) \subseteq \mathbb{R}^+$. The *infinitesimal subgroup* of (H, H^+, u) is

$$\inf(H) = \{a \in H : \phi(a) = 0 \text{ for all state } \phi\}.$$

It is not difficult to show that $\inf(H)$ does not depend on the order unit.
The quotient group $H/\inf(H)$ of a simple dimension group (H, H^+) is also a simple dimension group with positive cone

$$(H/\inf(H))^+ = \{[a] : a \in H^+\}.$$

For more details about ordered groups and dimension groups we refer to [10] and [17].

Lemma 1. *Let (H, H^+) be a simple dimension group equals to the direct limit*

$$\lim_{\to n}(\mathbb{Z}^{k_n}, M_n) = \mathbb{Z}^{k_0} \xrightarrow{M_0} \mathbb{Z}^{k_1} \xrightarrow{M_1} \mathbb{Z}^{k_2} \xrightarrow{M_2} \cdots .$$

Then for every $z = (z_n)_{n\geq 0}$ in the inverse limit

$$\lim_{\leftarrow n}((\mathbb{R}^+)^{k_n}, M_n^T) = (\mathbb{R}^+)^{k_0} \xleftarrow{M_0^T} (\mathbb{R}^+)^{k_1} \xleftarrow{M_1^T} (\mathbb{R}^+)^{k_2} \xleftarrow{M_2^T} \cdots ,$$

the function $\phi_z : H \to \mathbb{R}$ given by $\phi([n, v]) =< v, z_n >$, for every $[n, v] \in H$, is well defined and is a homomorphism of groups such that $\phi_z(H^+) \subseteq \mathbb{R}^+$. Conversely, for every group homomorphism $\phi : H \to \mathbb{R}$ such that $\phi(H^+) \subseteq \mathbb{R}^+$, there exists a unique $z \in \lim_{\leftarrow n}((\mathbb{R}^+)^{k_n}, M_n^T)$ such that $\phi = \phi_z$.

Proof. Let $z \in \varprojlim_n((\mathbb{R}^+)^{k_n}, M_n^T)$. If $(m, w) \in [n, v]$ then there exists $k \geq m, n$ such that $M_k \cdots M_m w = M_k \cdots M_n v$. Then

$$\langle v, z_n\rangle = v^T M_n^T \cdots M_k^T z_{k+1} = w^T M_m^T \cdots M_k^T z_{k+1} = w^T z_m = \langle w, z_m\rangle.$$

This implies that ϕ_z is well defined. If $[n, v]$ is in H^+, we can assume that $v \geq 0$, then $\phi_z([n, v]) = \langle v, z_n\rangle \geq 0$. It is straightforward to show that ϕ_z is a group homomorphism so that $\phi_z(H^+) \subseteq \mathbb{R}^+$.
Convesly, let $\phi : H \to \mathbb{R}$ be a group homomorphism such that $\phi(H^+) \subseteq \mathbb{R}^+$. For every $n \geq 0$, the function $\phi_n : \mathbb{Z}^{k_n} \to \mathbb{R}$ such that $\phi_n(v) = \phi([n, v])$, is a group homomorphism verifying $\phi_n((\mathbb{Z}^+)^{k_n}) \subseteq \mathbb{R}^+$. Then $z_n = (\phi_n(e_1), \cdots, \phi_n(e_{k_n})) \geq 0$, where $e_1, \cdots, e_{k_n}$ are the unitary vectors in $\mathbb{Z}^{k_n}$. We have $\phi([n, v]) = \phi_n(v) = \langle v, z_n\rangle$, for every $v \in \mathbb{Z}^{k_n}$ and $n \geq 0$. Thus if we show that $z = (z_n)_{n\geq 0}$ is in $\varprojlim_n((\mathbb{R}^+)^{k_n}, M_n^T)$, we get $\phi = \phi_z$ for z in the inverse limit. But this is direct because $\phi_n(v) = \phi_{n+1}(M_n v)$, for every $v \in \mathbb{Z}^{k_n}$. Applying this to the unitary vectors, we get $z_n = M_n^T z_{n+1}$, for every $n \geq 0$. Different elements in $\varprojlim_n((\mathbb{R}^+)^{k_n}, M_n^T)$ define different homomorphisms, so z is unique. □

2.5. **Associated ordered group and orbit equivalence.** Let (X, T, G) be a topological dynamical system such that X is a Cantor set and T is minimal. The ordered group associated to (X, T, G) is the unital ordered group

$$\mathcal{G}(X, T, G) = (D_m(X, T, G), D_m(X, T, G)^+, [1]),$$

where

$$D_m(X, T, G) = C(X, \mathbb{Z})/\{f \in C(X, \mathbb{Z}) : \int f d\mu = 0, \forall \mu \in \mathcal{M}(X, T, G)\},$$

$$D_m(X, T, G)^+ = \{[f] : f \geq 0\},$$

and $[1] \in D_m(X, T, G)$ is the class of the constant function 1.

Two topological dynamical systems (X_1, T_1, G_1) and (X_2, T_2, G_2) are (topologically) *orbit equivalent* if there exists a homeomorphism $F : X_1 \to X_2$ such that $F(o_{T_1}(x)) = o_{T_2}(F(x))$ for every $x \in X_1$.
Theorem 2.5 in [14] implies that for every integer $d \geq 1$, the isomorphic class of $\mathcal{G}(X, T, \mathbb{Z}^d)$ is a total invariant of the topological orbit equivalence class of $(X, T, \mathbb{Z}^d)$ for minimal $\mathbb{Z}^d$ action on a Cantor set.

3. Characterization of Choquet simplices

A compact, convex, and metrizable subset K of a locally convex real vector space is said to be a (metrizable) Choquet simplex, if for each $v \in K$ there is a unique probability measure μ supported on the set of extreme points of K such that $\int x d\mu(x) = v$.

In this section we show that any metrizable Choquet simplex is affine homeomorphic to an inverse limit defined by sequences of matrices that we call *manageable.*
We say that a sequence of non-negative integer matrices $(M_n)_{n\geq 0}$ is *manageable* with respect to the increasing sequence of positive integers $(p_n)_{n\geq 0}$, if for every $n \geq 0$ the integer p_n divides p_{n+1}, and if the matrix M_n verifies the following properties:

(1) M_n has $k_n \geq 3$ rows and $k_{n+1} \geq 3$ columns.
(2) $\sum_{i=1}^{k_n} M_n(i, k) = \frac{p_{n+1}}{p_n}$, for every $1 \leq k \leq k_{n+1}$,
(3) $k_{n+1} \leq \min\{M_n(i, k) : 1 \leq i \leq k_n, 1 \leq k \leq k_{n+1}\}$.

Let p be a positive integer. For every $n \geq 1$ we denote by $\triangle(n, p)$ the closed convex hull generated by the vectors $\frac{1}{p}e_1^{(n)}, \cdots, \frac{1}{p}e_n^{(n)}$, where $e_1^{(n)}, \cdots, e_n^{(n)}$ is the canonical base in $\mathbb{R}^n$. Thus $\triangle(n, 1)$ is the unitary simplex in $\mathbb{R}^n$.
Observe that if $(M_n)_{n\geq 0}$ is a manageable sequence of matrices with respect to $(p_n)_{n\geq 0}$, then for each $n \geq 0$, the map $M_n : \triangle(k_{n+1}, p_{n+1}) \to \triangle(k_n, p_n)$ is well defined, where k_n is the number of rows of M_n.

3.1. Finite dimensional Choquet simplices.

For technical reasons, we have to separate the finite and the infinite dimensional cases.

Lemma 2. *Let K be a finite dimensional metrizable Choquet simplex with exactly $d \geq 1$ extreme points. Let $(p_n)_{n\geq 0}$ be an increasing sequence of positive integers such that for every $n \geq 0$ the integer p_n divides p_{n+1}, and let $k \geq \max\{3, d\}$. Then there exist an increasing subsequence $(n_i)_{i\geq 0}$ of indices and a sequence $(M_i)_{i\geq 0}$ of square k-dimensional matrices which is manageable with respect to $(p_{n_i})_{i\geq 0}$ and such that K is affine homeomorphic to $\varprojlim_n(\triangle(k, p_{n_i}), M_i)$.*

Proof. Let $k \geq \max\{3, d\}$, we will define the subsequence $(n_i)_{i\geq 0}$ by induction on i through a condition explained later. For every $i \geq 0$, we define M_i the k-dimensional matrix by

$$M_i(l, j) = \begin{cases} \frac{p_{n_{i+1}}}{p_{n_i}} - k(k-1) & \text{if} \quad 1 \leq l = j \leq d \\ k & \text{if} \quad l \neq j, 1 \leq l \leq k \text{ and } 1 \leq j \leq d \end{cases}$$

and

$$M_i(\cdot, j) = M_i(\cdot, d) \text{ for } d < j \leq k.$$

By the very definition, M_i is a positive matrix having $k \geq 3$ rows and columns ; $\sum_{l=1}^{k} M_i(l, j) = \frac{p_{n_{i+1}}}{p_{n_i}}$ for every $1 \leq j \leq k$ and the range of M_i is at most d. Thus the convex set $\varprojlim_n(\triangle(k, p_{n_i}), M_i)$ has at most d extreme points.

If it has exactly d extreme points, it is affine homeomorphic to K. Thus by the Hadamard's Lemma, it is enough to ensure that the submatrices $(M_0 \dots M_i(l,j))_{1\le l,j\le d}$ are diagonally strictly dominant matrices.
Let $n_0 = 0$ and n_1 be an integer so that

$$\frac{p_{n_1}}{p_0} - k(k-1) \ge k(d-1)+1 \ge k.$$

This ensures that the matrix $(M_0(l,j))_{1\le l,j\le d}$ is diagonally strictly dominant.
Now let us assume that the sequence $n_0, \dots, n_{i-1}$ is defined so that $(M_0 \dots M_{i-1}(l,j))_{1\le l,j\le d}$ is a diagonally strictly dominant matrix. Then standard calculus show that for any $l, j \in \{1, \dots, d\}$,

$$M_0 \dots M_i(l,j) = (\frac{p_{n_{i+1}}}{p_{n_i}} - k(k-1))M_0 \dots M_{i-1}(l,j) + b_{l,j},$$

where the coefficient $b_{l,j}$ does not depend on the diagonal coefficient $\frac{p_{n_{i+1}}}{p_{n_i}} - k(k-1)$. The diagonal is dominant means that for any $l \in \{1, \dots, d\}$

$$(\frac{p_{n_{i+1}}}{p_{n_i}} - k(k-1))M_0 \dots M_{i-1}(l,l) + b_{l,l} \quad > \quad \sum_{j=1, j\neq l}^{d} (\frac{p_{n_{i+1}}}{p_{n_i}} - k(k-1))M_0 \dots M_{i-1}(l,j) + b_{l,j}.$$

Now, let n_{i+1} be an integer so that

$$\frac{p_{n_{i+1}}}{p_{n_i}} - k(k-1) > \max\left(\sup_{1\le l\le d} \sum_{j=1,j\neq l}^{d} b_{l,j} - b_{l,l}, k\right);$$

since $M_0 \dots M_{i-1}(l,l) - \sum_{j=1,j\neq l}^{d} M_0 \dots M_{i-1}(l,j) \ge 1$, the former inequality is true, and the matrix $(M_0 \dots M_i(l,j))_{1\le l,j\le d}$ is diagonally strictly dominant.
It is straightforward to check that $(M_i)_i$ is manageable with respect to $(p_{n_i})_i$. □

3.2. **Infinite dimensional Choquet simplices.** Firt, we use the following characterization of infinite dimensional metrizable Choquet simplex.

Lemma 3 ([21], Corollary p.186)**.** *For every infinite dimensional metrizable Choquet simplex K, there exists a sequence of matrices $(A_n)_{n\ge1}$ such that for every $n \ge 1$*

(1) $A_n : \triangle(n+1,1) \to \triangle(n,1)$ is well defined and surjective,
(2) K is affine homeomorphic to $\varprojlim_n (\triangle(n,1), A_n)$.

Our strategy is to approximate the sequence of matrices $(A_n)_n$ by a manageable sequence. Then we show that the associated inverse limits are affine homeomorphic. For this, we need the following classical density result.

Lemma 4. *Let $\mathbf{r} = (r_n)_{n\ge0}$ be a sequence of integers such that $r_n \ge 2$ for every $n \ge 0$. Let $C_{\mathbf{r}}$ be the subgroup of $(\mathbb{R}, +)$ generated by $\{(r_0 \cdots r_n)^{-1} : n \ge 0\}$. Then*

$$(C_{\mathbf{r}})^p \cap \triangle(p,1) \cap \{v \in \mathbb{R}^p : v > 0\}$$

is dense in $\triangle(p,1)$, for every $p \ge 2$.

Proof. Suppose that $p = 2$. Let $m \ge 1$ and $\Gamma_m = \mathbb{Z}(r_0 \cdots r_m)^{-1}$. For $\alpha = (\alpha_1, \alpha_2) \in \triangle(2,1) \setminus (\Gamma_m)^2$, let $k_1, k_2 \ge 0$ be the integers such that

$$\frac{k_1}{r_0 \cdots r_m} \le \alpha_1 < \frac{k_1}{r_0 \cdots r_m} + \frac{1}{r_0 \cdots r_m}$$

and

$$\frac{k_2}{r_0 \cdots r_m} \le \alpha_2 < \frac{k_2}{r_0 \cdots r_m} + \frac{1}{r_0 \cdots r_m}.$$

Since there is $i \in \{1,2\}$ such that $\alpha_i \notin \Gamma_m$, we have

$$1 - \frac{1}{r_0 \cdots r_m} < \frac{k_1}{r_0 \cdots r_m} + \frac{k_2}{r_0 \cdots r_m} + \frac{1}{r_0 \cdots r_m} < 1 + \frac{1}{r_0 \cdots r_m},$$

which implies that

$$\frac{k_1}{r_0 \cdots r_m} + \frac{k_2}{r_0 \cdots r_m} + \frac{1}{r_0 \cdots r_m} = 1.$$

Thus $v = (r_0 \cdots r_m)^{-1}(k_1, k_2 + 1)$ and $w = (r_0 \cdots r_m)^{-1}(k_1 + 1, k_2)$ are elements in $(\Gamma_m)^2 \cap \triangle(2,1)$ such that $\|v - \alpha\|_1, \|w - \alpha_1\|_1 \leq 2(r_0 \cdots r_m)^{-1}$. Furthermore, v or w is strictly positive. If α is in $(\Gamma_m)^2$ and one of its coordinates is not positive, then $\alpha = e_1^{(2)}$ or $\alpha = e_2^{(2)}$. Observe that $(1 - (r_0 \cdots r_m)^{-1}, (r_0 \cdots r_m)^{-1})$ and $((r_0 \cdots r_m)^{-1}, 1 - (r_0 \cdots r_m)^{-1})$ are positive elements in $(\Gamma_m)^2 \cap \triangle(2,1)$ at distance at most $2(r_0 \cdots r_m)^{-1}$ from α. Since m can be arbitrarily large, we conclude that the lemma is true for $p = 2$

For $p > 2$ we conclude by induction on p. Indeed, if $\alpha = (\alpha_i)_{i=1}^p \in \triangle(p,1)$ then $\beta = (\alpha_1, \cdots, \alpha_{p-2}, \alpha_{p-1} + \alpha_p)$ is in $\triangle(p-1,1)$. Thus for $\varepsilon > 0$ there exists a strictly positive element $z = (z_1, \cdots, z_{p-1})$ in $(C_{\mathbf{r}})^{p-1} \cap \triangle(p-1,1)$ such that $\|z - \beta\|_1 < \varepsilon$. Since $C_{\mathbf{r}}$ is dense in $\mathbb{R}$, there exists $a \in C_{\mathbf{r}}$ such that $0 < a < z_{p-1}$. If $\alpha_p < \alpha_p + \alpha_{p-1}$, by taking ε small enough, we can assume that $\alpha_p < z_{p-1}$. In this case we can choose a in order that $|a - \alpha_p| < \varepsilon$. If $\alpha_p = \alpha_p + \alpha_{p-1}$, then we can choose a in order that $|\alpha_p - a| \leq 2\varepsilon$. In any case, we get that $w = (z_1, \cdots, z_{p-2}, z_{p-1} - a, a)$ is a strictly positive element in $(C_{\mathbf{r}})^p \cap \triangle(p,1)$ such that $\|w - \alpha\|_1 < 5\varepsilon$. □

Lemma 5. *Let K be an infinite dimensional metrizable Choquet simplex, and let $(p_n)_{n\geq 0}$ be an increasing sequence of positive integers such that for every $n \geq 0$ the integer p_n divides p_{n+1}. Then there exist an increasing subsequence $(n_i)_{i\geq 1}$ of indices and a manageable sequence $(M_i)_{i\geq 1}$ of matrices with respect to $(p_{n_i})_{i\geq 0}$ such that K is affine homeomorphic to the inverse limit $\varprojlim_n (\triangle(k_i, p_{n_i}), M_i)$, where k_i is the number of rows of M_i, for every $i \geq 0$.*

Proof. For every $n \geq 0$, let $r_n \geq 2$ be the integer such that $p_{n+1} = p_n r_n$.
Let $(A_n)_{n\geq 1}$ be the sequence of matrices given in Lemma 3. We can assume that $A_{n+2} : \triangle(n+3,1) \longrightarrow \triangle(n+2,1)$, for every $n \geq 1$. Now we define the subsequence $(n_i)_i$ by induction.

We set $n_1 = 0$.

Let $i \geq 1$ and suppose that we have defined $n_i \geq 0$. We set $\mathbf{r^{(i)}} = (r_n)_{n\geq n_i}$. For every $1 \leq j \leq i+3$, Lemma 4 ensures the existence of $v^{(i,j)} \in (C_{\mathbf{r^{(i)}}})^{i+2} \cap \triangle(i+2,1) \cap \{v \in \mathbb{R}^{i+2} : v > 0\}$ such that

$$\|v^{(i,j)} - A_{i+2}(\cdot, j)\|_1 < \frac{1}{2^i}. \tag{1}$$

Let B_i be the matrix given by

$$B_i(\cdot, j) = v^{(i,j)}, \text{ for every } 1 \leq j \leq i+3.$$

Observe that (1) implies that

$$\sum_{n\geq 1} \sup\{\|A_n v - B_n v\|_1 : v \in \triangle_{n+3}\} < \infty.$$

It follows from [5, Lemma 9] that K is affine homeomorphic $\varprojlim_n (\triangle(i+2,1), B_i)$.

Let $n_{i+1} > n_i$ be such that $r_{n_i} \cdots r_{n_{i+1}-1} v^{(i,j)}$ is an integer vector and such that $r_{n_i} \cdots r_{n_{i+1}-1} v^{(i,j)} > i+3$, for every $1 \leq j \leq i+3$.

We define

$$M_i = \frac{p_{n_{i+1}}}{p_{n_i}} B_i.$$

Thus $M_i = P_i^{-1} B_i P_{i+1}$, where P_i is the diagonal matrix given by $P_i(j,j) = p_{n_i}$ for every $1 \leq j \leq i+2$ and $i \geq 1$. This shows that $\varprojlim_n (\triangle(i+2,1), B_i)$ is affine homeomorphic to $\varprojlim_n (\triangle(i+2, p_{n_i}), M_i)$.

The proof conclude verifying that $(M_i)_{i\geq 0}$ is manageable with respect to $(p_{n_i})_{i\geq 0}$. □

4. Suitable Følner sequences and connected components.

Let G be a residually finite group, and let $(\Gamma_n)_{n\geq 0}$ be a nested sequence of finite index normal subgroup of G such that $\bigcap_{n\geq 0} \Gamma_n = \{e\}$.

For technical reasons it is important to notice that since the groups Γ_n are normal, we have $g\Gamma_n = \Gamma_n g$, for every $g \in G$.

To construct a Toeplitz G-subshift that is an almost 1-1 extension of the odometer defined by the sequence $(\Gamma_n)_n$, we need a "suitable" sequence $(F_n)_n$ of fundamental domains of G/Γ_n. More precisely, each F_{n+1} has to be tileable by translated copies of F_n. To control the simplex of invariant measure of the subshift, we need in addition the sequence $(F_n)_n$ to be Følner. We did not find in the specialized litterature a result ensuring these conditions.

4.1. **Suitable sequence of fundamental domains.**

Lemma 6. *Let $(D_n)_{n\geq 0}$ be an increasing sequence of finite subsets of G such that for every $n \geq 0$, $e \in D_n$ and D_n is a fundamental domain of G/Γ_n. Let $(n_i)_{i\geq 0} \subseteq \mathbb{Z}^+$ be an increasing sequence. Consider $(F_i)_{i\geq 0}$ defined by $F_0 = D_{n_0}$ and*

$$F_i = \bigcup_{v \in D_{n_i} \cap \Gamma_{n_{i-1}}} vF_{i-1} \text{ for every } i \geq 1.$$

Then for every $i \geq 0$ we have the following:

(1) $F_i \subseteq F_{i+1}$ and F_i is a fundamental domain of G/Γ_{n_i}.
(2) $F_{i+1} = \bigcup_{v \in F_{i+1} \cap \Gamma_{n_i}} vF_i$.

Proof. Since $e \in D_{n_i}$, the sequence $(F_i)_{i\geq 0}$ is increasing.

$F_0 = D_{n_0}$ is a fundamental domain of G/Γ_{n_0}. We will prove by induction on i that F_i is a fundamental domain of G/Γ_{n_i}. Let $i > 0$ and suppose that F_{i-1} is a fundamental domain of $G/\Gamma_{n_{i-1}}$.

Let $v \in D_{n_i}$. There exist then $u \in F_{i-1}$ and $w \in \Gamma_{n_{i-1}}$ such that $v = wu$. Let $z \in D_{n_i}$ and $\gamma \in \Gamma_{n_i}$ be such that $w = \gamma z$. Since $z \in \Gamma_{n_{i-1}} \cap D_{n_i}$ and $v = \gamma z u$, we conclude that F_i contains one representing element of each class in G/Γ_{n_i}.

Let $w_1, w_2 \in F_i$ be such that there exists $\gamma \in \Gamma_{n_i}$ verifying $w_1 = \gamma w_2$. By definition, $w_1 = v_1 u_1$ and $w_2 = v_2 u_2$, for some $u_1, u_2 \in F_{i-1}$ and $v_1, v_2 \in D_{n_i} \cap \Gamma_{n_{i-1}}$. This implies that u_1 and u_2 are in the same class of $G/\Gamma_{n_{i-1}}$. Since F_{i-1} is a fundamental domain, we have $u_1 = u_2$. From this we get $v_1 = \gamma v_2$, which implies that $v_1 = v_2$. Thus we deduce that F_i contains at most one representing element of each class in G/Γ_{n_i}. This shows that F_i is a fundamental domain of G/Γ_{n_i}.

The neutral element e is contained in $D_0 = F_0 \subseteq F_{i-1}$. Then by definition of F_i we have $ve \in F_i$ for every $v \in D_{n_i} \cap \Gamma_{n_{i-1}}$. This shows that $D_{n_i} \cap \Gamma_{n_{i-1}} \subseteq F_i \cap \Gamma_{n_{i-1}}$. For $v \in F_i \cap \Gamma_{n_{i-1}}$, let $u \in F_{i-1}$ and $\gamma \in D_{n_i} \cap \Gamma_{n_{i-1}}$ be such that $v = \gamma u$. Since v and γ are in $\Gamma_{n_{i-1}}$, we have $u \in \Gamma_{n_{i-1}}$, which implies that $u = e$ and $v = \gamma \in D_{n_i} \cap \Gamma_{n_{i-1}}$. □

In this paper, by Følner sequences we mean right Følner sequences. That is, a sequence $(F_n)_{n\geq 0}$ of nonempty finite sets of G is a Følner sequence if for every $g \in G$

$$\lim_{n\to\infty} \frac{|F_n g \triangle F_n|}{|F_n|} = 0.$$

Observe that $(F_n)_{n\geq 0}$ is a right Følner sequence if and only if $(F_n^{-1})_{n\geq 0}$ is a left Følner sequence.

Lemma 7. *Suppose that G is amenable. There exists an increasing sequence $(n_i)_{i\geq 0} \subseteq \mathbb{Z}^+$ and a Følner sequence $(F_i)_{i\in\mathbb{Z}^+}$, such that*

i) $F_i \subseteq F_{i+1}$ and F_i is a fundamental domain of G/Γ_{n_i}, for every $i \geq 0$.
ii) $G = \bigcup_{i\geq 0} F_i$.
iii) $F_{i+1} = \bigcup_{v\in F_{i+1}\cap\Gamma_{n_i}} vF_i$, for every $i \geq 0$.

Proof. From [20, Proposition 4.1], there exists an increasing sequence $(m_i)_{i\geq 0} \subseteq \mathbb{Z}^+$ and a Følner sequence $(D_i)_{i\in\mathbb{Z}^+}$ such that for every $i \geq 0$, $D_i \subseteq D_{i+1}$, D_i is a fundamental domain of G/Γ_{m_i}, and $G = \bigcup_{i\geq 0} D_i$. Up to take subsequences, we can assume that D_i is a fundamental domain of G/Γ_i, for every $i \geq 0$, and that $e \in D_0$.
We will construct the sequences $(n_i)_{i\geq 0}$ and $(F_n)_{n\geq 0}$ as follows:

Step 0*:* We set $n_0 = 0$ and $F_0 = D_0$.

Step i: Let $i > 0$. We assume that we have chosen n_j and F_j for every $0 \leq j < i$. We take $n_i > n_{i-1}$ in order that the following two conditions are verified:

$$\frac{|D_{n_i} g \triangle D_{n_i}|}{|D_{n_i}|} < \frac{1}{i|F_{i-1}|}, \text{ for every } g \in F_{i-1}. \tag{2}$$

$$D_{n_{i-1}} \subseteq \bigcup_{v\in D_{n_i}\cap\Gamma_{n_{i-1}}} vF_{i-1}. \tag{3}$$

Such integer n_i exists because $(D_n)_{n\geq 0}$ is a Følner sequence and F_{i-1} is a fundamental domain of $G/\Gamma_{n_{i-1}}$ (then $G = \bigcup_{v\in\Gamma_{n_{i-1}}} vF_{i-1}$).
We define

$$F_i = \bigcup_{v\in D_{n_i}\cap\Gamma_{n_{i-1}}} vF_{i-1}.$$

Lemma 6 ensures that $(F_i)_{i\geq 0}$ verifies *i)* and *iii)* of the lemma. The equation (3) implies that $(F_i)_{i\geq 0}$ verifies *ii)* of the lemma.
It remains to show that $(F_i)_{i\geq 0}$ is a Følner sequence.
By definition of F_i we have

$$F_i \setminus D_{n_i} \subseteq \bigcup_{g\in F_{i-1}} D_{n_i} g \setminus D_{n_i}.$$

Then by equation (2) we get

$$\begin{aligned} \frac{|F_i \setminus D_{n_i}|}{|D_{n_i}|} &\leq \sum_{g\in F_{i-1}} \frac{|D_{n_i} g \setminus D_{n_i}|}{|D_{n_i}|} \\ &\leq |F_{i-1}| \frac{1}{i|F_{i-1}|} = \frac{1}{i}. \end{aligned}$$

Since

$$|F_i \cap D_{n_i}| + |D_{n_i} \setminus F_i| = |D_{n_i}| = |F_i| = |F_i \cap D_{n_i}| + |F_i \setminus D_{n_i}|,$$

we obtain

$$\frac{|D_{n_i} \setminus F_i|}{|D_{n_i}|} \leq \frac{1}{i}.$$

Let $g \in G$. Since

$$\begin{array}{rcl} F_i g \setminus F_i & = & (F_i \cap D_{n_i})g \setminus F_i \cup (F_i \setminus D_{n_i})g \setminus F_i \\ & \subseteq & (F_i \cap D_{n_i})g \setminus F_i \cup (F_i \setminus D_{n_i})g \\ & \subseteq & D_{n_i}g \setminus (F_i \cap D_{n_i}) \cup (F_i \setminus D_{n_i})g, \end{array}$$

we have

$$\frac{|F_i g \setminus F_i|}{|F_i|} \leq \frac{|D_{n_i}g \setminus (F_i \cap D_{n_i})|}{|D_{n_i}|} + \frac{|(F_i \setminus D_{n_i})g|}{|D_{n_i}|} \leq \frac{|D_{n_i}g \setminus (F_i \cap D_{n_i})|}{|D_{n_i}|} + \frac{1}{i}. \tag{4}$$

On the other hand, the relation

$$D_{n_i}g \setminus D_{n_i} = D_{n_i}g \setminus ((D_{n_i} \cap F_i) \cup D_{n_i} \setminus F_i) = (D_{n_i}g \setminus (D_{n_i} \cap F_i)) \setminus (D_{n_i} \setminus F_i),$$

implies that

$$\begin{array}{rcl} D_{n_i}g \setminus (F_i \cap D_{n_i}) & = & [(D_{n_i}g \setminus (F_i \cap D_{n_i})) \cap (D_{n_i} \setminus F_i)] \cup (D_{n_i}g \setminus (F_i \cap D_{n_i})) \setminus (D_{n_i} \setminus F_i) \\ & = & [(D_{n_i}g \setminus (F_i \cap D_{n_i})) \cap (D_{n_i} \setminus F_i)] \cup D_{n_i}g \setminus D_{n_i} \\ & \subseteq & D_{n_i} \setminus F_i \cup D_{n_i}g \setminus D_{n_i}, \end{array}$$

which ensures that

$$\frac{|D_{n_i}g \setminus (F_i \cap D_{n_i})|}{|D_{n_i}|} \leq \frac{|D_{n_i} \setminus F_i|}{|D_{n_i}|} + \frac{|D_{n_i}g \setminus D_{n_i}|}{|D_{n_i}|}. \tag{5}$$

From equations (4) and (5), we obtain

$$\frac{|F_i g \setminus F_i|}{|F_i|} \leq \frac{2}{i} + \frac{|D_{n_i}g \setminus D_{n_i}|}{|D_{n_i}|},$$

which implies

$$\lim_{i \to \infty} \frac{|F_i g \setminus F_i|}{|F_i|} = 0. \tag{6}$$

In a similar way we deduce that

$$F_i \setminus F_i g \subseteq (D_{n_i} \setminus (F_i \cap D_{n_i})g) \cup F_i \setminus D_{n_i},$$

$$D_{n_i} \setminus D_{n_i}g = (D_{n_i} \setminus (D_{n_i} \cap F_i)g) \setminus (D_{n_i} \setminus F_i),$$

and

$$D_{n_i} \setminus (F_i \cap D_{n_i})g \subseteq D_{n_i} \setminus F_i \cup D_{n_i} \setminus D_{n_i}g.$$

Combining the last three equations we get

$$\frac{|F_i \setminus F_i g|}{|F_i|} \leq \frac{2}{i} + \frac{|D_{n_i} \setminus D_{n_i}g|}{|D_{n_i}|},$$

which implies

$$\lim_{i \to \infty} \frac{|F_i \setminus F_i g|}{|F_i|} = 0. \tag{7}$$

Equations (6) and (7) imply that $(F_i)_{i \geq 0}$ is Følner. □

The following result is a direct consequence of Lemma 7.

Lemma 8. *Let G be an amenable residually finite group and let $(\Gamma_n)_{n\geq 0}$ be a decreasing sequence of finite index normal subgroups of G such that $\bigcap_{n\geq 0}\Gamma_n=\{e\}$. There exists an increasing sequence $(n_i)_{i\geq 0}\subseteq\mathbb{Z}^+$ and a Følner sequence $(F_i)_{i\geq 0}$ of G such that*

1. *$\{e\}\subseteq F_i\subseteq F_{i+1}$ and F_i is a fundamental domain of G/Γ_{n_i}, for every $n\geq 0$.*
2. *$G=\bigcup_{i\geq 0}F_i$.*
3. *$F_j=\bigcup_{v\in F_j\cap\Gamma_{n_i}}vF_i$, for every $j>i\geq 0$.*

Proof. The existence of the sequence of subgroups of G and the Følner sequence verifying (1), (2) and (3) for $j=i+1$ is direct from Lemma 7. Using induction, it is straightforward to show (3) for every $j>i\geq 0$. □

4.2. **Connected components.** We recall here the notion of connected component of a discrete group G. This notion will be usefull to define a Toeplitz sequence. Let Γ be a finitely generated subgroup of G. Let S be a symmetric generating set of Γ and let $C\subseteq\Gamma$ be a non empty set. The *connected components* of C with respect to S are the equivalence classes of the following equivalence relation defined on C:

$$g\sim h \iff \text{there exist } s_1,\cdots,s_k\in S\cup\{e\}, \text{ such that for every } 1\leq i\leq k,$$
$$gs_1\cdots s_i\in C \text{ and } gs_1\cdots s_k=h.$$

The set C is said to be *connected* if C has only one connected component.

Lemma 9. *Let $C\subseteq\Gamma$ be a connected set. For every finite number $1\leq r\leq|C|$ there exists a connected subset C' of C such that $|C'|=r$.*

Proof. Let $1\leq r\leq|C|$ be a finite number. If $r=1$ then $C'=\{g\}$, for $g\in C$, is connected. Suppose there exists a connected subset C'' of C such that $|C''|=r-1$. Since $|C|\geq r$, we can choose $h\in C\setminus C''$. Due to C is connected, for $g\in C''$ there exist $s_1,\cdots,s_k\in S$ such that for every $1\leq i\leq k$, $gs_1\cdots s_i\in C$ and $gs_1\cdots s_k=h$. Let $g\in C''$ and $h'=gs_1\cdots s_i$ be such that $i=\min\{1\leq j\leq k: gs_1\cdots s_j\notin C''\}$. The set $C'=C''\cup\{h'\}$ is connected and has exactly r elements. □

Let us recall that for a group G finitely generated and Γ_n a subgroup of finite index, then Γ_n is finitely generated (see for example [1, Proposition 6.6.2]). Thus for $(\Gamma_n)_{n\geq 0}$ and $(F_n)_{n\geq 0}$ be as in Lemma 8.

Definition 1. *For every $n\geq 0$, let S_n be a finite symmetric generating subset of Γ_n.*

Since the sequence $(F_n)_{n\geq 0}$ is increasing and the union of these sets covers G, there exists a subsequence $(F_{n_i})_{i\geq 0}$ such that e and the elements of S_{n_i} are in the same connected component of $F_{n_{i+1}}\cap\Gamma_{n_i}$ with respect to S_{n_i}.

5. PROOF OF THEOREM A.

Let G be an amenable finitely generated residually finite group. Let $(\Gamma_n)_{n\geq 0}$ and $(F_n)_{n\geq 0}$ be as in Lemma 8.
For every $n\geq 0$, we call R_n the set $[(F_n\cup F_n^{-1})F_n^{-1}\cup S_n]\cup[(F_n\cup F_n^{-1})F_n^{-1}\cup S_n]^{-1}$. This will enable us to define a "border" of each domain F_{n+1}.
Let Σ be a finite alphabet. For every $n\geq 0$, let $k_n\geq 3$ be an integer and let $\{B_{n,1},\cdots,B_{n,k_n}\}\subseteq\Sigma^{F_n}$ be a collection of different functions. We say that $(\{B_{n,1},\cdots,B_{n,k_n}\})_{n\geq 0}$ verifies conditions (C1)-(C4) if it verifies the following four conditions for any $n\geq 0$:

- (C1) $\sigma^{\gamma^{-1}}(B_{n+1,k})|_{F_n}\in\{B_{n,i}:1\leq i\leq k_n\}$, for every $\gamma\in F_{n+1}\cap\Gamma_n$, $1\leq k\leq k_{n+1}$.
- (C2) $B_{n+1,k}|_{F_n}=B_{n,1}$, for every $1\leq k\leq k_{n+1}$.

(C3) For any $g \in F_n$ such that for some $1 \leq k, k' \leq k_n$, $B_{n,k}(gv) = B_{n,k'}(v)$ for all $v \in F_n \cap F_n g^{-1}$, then $g = e$.

(C4) $\sigma^{\gamma^{-1}}(B_{n+1,k})|_{F_n} = B_{n,k_n}$ for every $\gamma \in (F_{n+1} \cap \Gamma_n) \cap F_{n+1} \setminus F_{n+1}g^{-1}$, for some $g \in R_n$.

In this case, for every $n \geq 0$ we define the matrix $M_n \in \mathcal{M}_{k_n \times k_{n+1}}(\mathbb{Z}^+)$ as

$$M_n(i,k) = |\{v \in F_{n+1} \cap \Gamma_n : B_{n+1,k}|_{vF_n} = B_{n,i}\}|.$$

In the next lemma, we show that conditions (C1) and (C2) are sufficient to construct a Toeplitz sequence. The technical conditions (C3) (aperiodicity) and (C4) (also known as "forcing the border") will ensure the existence of a good sequence of partitions $(\mathcal{P}_n)_n$ spanning the topology. This will allow to give a characterization of the set of invariant probability measures and the ordered group of the associated Toeplitz subshift.

Lemma 10. *Let $(\{B_{n,1}, \cdots, B_{n,k_n}\})_{n\geq 0}$ be a sequence that verifies conditions (C1)-(C4). Then:*

1. *The set $\bigcap_{n\geq 0}[B_{n,1}]$ contains only one element x_0 which is a Toeplitz sequence.*
2. *Let X be the orbit closure of x_0 with respect to the shift action. For every $n \geq 0$,*
$$\mathcal{P}_n = \{\sigma^{u^{-1}}([B_{n,k}] \cap X) : 1 \leq k \leq k_n, u \in F_n\}$$
is a clopen partition of X. Moreover, $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$ and $(\mathcal{P}_n)_{n\geq 0}$ spans the topology of X.
3. *The Toeplitz subshift $(X, \sigma|_X, G)$ is an almost 1-1 extension of the odometer $O = \varprojlim_n (G/\Gamma_n, \pi_n)$.*
4. *There is an affine homeomorphism between the set of invariant probability measures of $(X, \sigma|_X, G)$ and the inverse limit $\varprojlim_n (\triangle(k_n, |F_n|), M_n)$.*
5. *The ordered group $\mathcal{G}(X, \sigma|_X, G)$ is isomorphic to $(H/inf(H), (H/inf(H))^+, u+inf(H))$, where $H = \varinjlim_n (\mathbb{Z}^{k_n}, M_n^T)$, $H^+ = \{[v,n] : v \geq 0, n \geq 0\}$ and $u = [|F_0|(1, \cdots, 1), 0]$.*

Proof. **1.** Condition (C2) implies that $\bigcap_{n\geq 0}[B_{n,1}]$ is non empty, and since $G = \bigcup_{n\geq 0} F_n$, there is only one element x_0 in this intersection. Let X be the orbit closure of x_0. For every $n \geq 0$ and $1 \leq k \leq k_n$, we denote $C_{n,k} = [B_{n,k}] \cap X$.

Let $n \geq 0$. Condition (C1) and (3) of Lemma 8 imply that $\sigma^{\gamma^{-1}}(B_{m,k})|_{F_n} \in \{B_{n,i} : 1 \leq i \leq k_n\}$, for every $m > n$, $1 \leq k \leq k_m$ and $\gamma \in \Gamma_n \cap F_m$. From this we deduce that $\sigma^{\gamma^{-1}}(x_0)|_{F_n} \in \{B_{n,i} : 1 \leq i \leq k_n\}$, for every $\gamma \in \Gamma_n$.

The condition (C2) implies then that $\sigma^{\gamma^{-1}}(x_0)|_{F_{n-1}} = B_{n-1,1}$ for any $\gamma \in \Gamma_n$. Thus, for $g \in G$ and n such that $g \in F_{n-1}$, we get $x_0(\gamma g) = B_{n-1,1}(g)$ for any $\gamma \in \Gamma_n$. This shows that x_0 is Toeplitz.

2. If g is any element in G, then there exist $u \in F_n$ and $\gamma \in \Gamma_n$ such that $g = \gamma u$. Thus $\sigma^{g^{-1}}(x_0) = \sigma^{u^{-1}}(\sigma^{\gamma^{-1}}(x_0)) \in \sigma^{u^{-1}}(C_{n,k})$, for some $1 \leq k \leq k_n$. It follows that

$$\mathcal{P}_n = \{\sigma^{u^{-1}}(C_{n,k}) : 1 \leq k \leq k_n, u \in F_n\}$$

is a clopen covering of X. Condition (C3) ensures that the *set of return times* of x_0 to $\bigcup_{k=1}^{k_n} C_{n,k}$, *i.e.* the set $\{g \in G : \sigma^{g^{-1}}(x_0) \in \bigcup_{k=1}^{k_n} C_{n,k}\}$, is Γ_n. This implies that $\mathcal{P}_n$ is a partition. From (C1) we have that $\mathcal{P}_{n+1}$ is finer than $\mathcal{P}_n$.

Since every $\mathcal{P}_n$ is a partition, for every $n \geq 0$ and every $x \in X$ there are unique $v_n(x) \in F_n$ and $1 \leq k_n(x) \leq k_n$ such that

$$x \in \sigma^{v_n(x)^{-1}}(C_{n,k_n(x)}).$$

The collection $(\mathcal{P}_n)_{n\geq 0}$ spans the topology of X if and only if $(v_n(x))_{n\geq 0} = (v_n(y))_{n\geq 0}$ and $(k_n(x))_{n\geq 0} = (k_n(y))_{n\geq 0}$ imply $x = y$.

Let $x, y \in X$ be two sequences such that $v_n(x) = v_n(y) = v_n$ and $k_n(x) = k_n(y)$ for every $n \geq 0$. Let $g \in G$ such that $x(g) \neq y(g)$.
We have then for any $n \geq 0$

$$\sigma^{v_n}(x)|_{F_n} = \sigma^{v_n}(y)|_{F_n} \in \{B_{n,i} : 1 \leq i \leq k_n\}.$$

And then

$$x|_{v_n^{-1}F_n} = y|_{v_n^{-1}F_n}.$$

Thus by definition, we get $g \notin v_n^{-1}F_n$ for any n. We can take n sufficiently large in order that $g \in F_{n-1}$.
Let $\gamma \in \Gamma_n$ and $u \in F_n$ such that $v_n(x)g = \gamma u$. Observe that $ug^{-1} \notin F_n$. Indeed, if $ug^{-1} \in F_n$, then the relation $v_n(x) = \gamma u g^{-1}$ implies $\gamma = e$, but in that case we get $v_n(x)g = u \in F_n$ which is not possible by hypothesis. By the condition (C1), there exists an index $1 \leq i \leq k_n$ such that $\sigma^{\gamma^{-1}}(\sigma^{v_n}(x))|_{F_n} = B_{n,i}$ and then

$$x(g) = \sigma^{\gamma^{-1}}\sigma^{v_n}(x)(\gamma^{-1}v_n g) = B_{n,i}(u).$$

Let $\gamma' \in \Gamma_{n-1} \cap F_n$ and $u' \in F_{n-1}$ such that $u = \gamma' u'$. Since $\gamma' u' g^{-1} = ug^{-1} \notin F_n$, we get $\gamma' \in F_n \setminus F_n g u'^{-1}$. This implies that $\gamma' \in F_n \setminus F_n w$, for $w = gu'^{-1} \in R_{n-1}$ and $B_{n,i}(u) = B_{n-1,k_{n-1}}(u')$ by the condition (C4). Thus $x(g) = B_{n-1,k_{n-1}}(u')$. The same argument implies that $y(g) = B_{n-1,k_{n-1}}(u') = x(g)$ and we obtain a contradiction.
This shows that $(\mathcal{P}_n)_{n\geq 0}$ spans the topology of X.

3. The map $\pi : X \to O$ given by $\pi(x) = ([v_n(x)]_n)_{n\geq 0}$ is well defined, is a factor map and verifies $\pi^{-1}(\pi(x_0)) = \{x_0\}$. This shows that $(X, \sigma|_X, G)$ is an almost 1-1 extension of O.

4. Since $(\mathcal{P}_n)_{n\geq 0}$ spans the topology of X, for every $\mu \in \mathcal{M}(X, \sigma|_X, G)$, the sequence $(\mu_n = (\mu(C_{n,1}), \cdots, \mu(C_{n,k_n}))_{n\geq 0}$ determines completely μ. Moreover, since

$$M_n(i,k) = |\{v \in F_{n+1} : \sigma^{v^{-1}}(C_{n+1,k}) \subseteq C_{n,i}\}|, \text{ for every } 1 \leq i \leq k_n, 1 \leq k \leq k_{n+1} \text{ and } n \geq 0,$$

the function $\mu \mapsto (\mu_n)_{n\geq 0}$, from $\mathcal{M}(X, \sigma|_X, G)$ to $\varprojlim_n(\triangle(k_n, |F_n|), M_n)$, is well defined and is an affine homeomorphism (with respect to the weak-star topology and the product topology). Indeed it is standard through the Følner sequence to show that a sequence in $\varprojlim_n(\triangle(k_n, |F_n|), M_n)$ define an unique G-invariant probability measure.

5. Let $\phi : H \to D_m(X, \sigma|_X, G)$ be the function given by $\phi([v, n]) = \sum_{k=1}^{k_n} v_i[1_{C_{n,k}}]$, for every $v = (v_1, \cdots, v_{k_n}) \in \mathbb{Z}^{k_n}$ and $n \geq 0$. It is easy to check that ϕ is a well defined homomorphism of groups that verifies and $\phi(H^+) \subseteq D_m(X, \sigma|_X, G)^+$. Since $(\mathcal{P}_n)_{n\geq 0}$ spans the topology of X, every function $f \in C(X, \mathbb{Z})$ is constant on every atom of $\mathcal{P}_m$, for some $m \geq 0$. This implies that ϕ is surjective. Lemma 1 and the previous point 4, imply that $Ker(\phi) = inf(H)$. Finally, ϕ induces a isomorphism $\widehat{\phi} : H/inf(H) \to D_m(X, \sigma|_X, G)$ such that $\widehat{\phi}((H/inf(H))^+) = D_m(X, \sigma|_X, G)^+$. It remains to show that $\phi([|F_0|(1, \cdots, 1), 0]) = \sum_k |F_0|[1_{C_{0,k}}] = [1_X]$. This is true because for any G-invariant measure μ, since $\mathcal{P}_0$ is a partition, we have

$$1 = \sum_{\substack{1\leq k\leq k_0 \\ u\in F_0}} \mu(\sigma^{u^{-1}}(C_{0,k})) = \sum_{1\leq k\leq k_0} |F_0|\mu(C_{0,k}).$$

□

The next result shows that, up to telescope a manageable sequence of matrices, it is possible to obtain a manageable sequence of matrices with sufficiently large coefficient to satisfy the conditions of Lemma 10.

Lemma 11. *Let $(M_n)_{n\geq 0}$ be a sequence of matrices manageable with respect to $(|F_n|)_{n\geq 0}$. Let k_n be the number of rows of M_n, for every $n \geq 0$.*
Then there exists an increasing sequence $(n_i)_{i\geq 0} \subseteq \mathbb{Z}^+$ such that for every $i \geq 0$ and every $1 \leq k \leq k_{n_{i+1}}$,

- (i) $R_{n_i} \subseteq F_{n_{i+1}}$,
- (ii) *For every* $1 \leq l \leq k_{n_i}$,
$$M_{n_i} M_{n_i+1} \cdots M_{n_{i+1}-1}(l,k) > |S_{n_i}| + |\bigcup_{g\in R_{n_i}} F_{n_{i+1}} \setminus F_{n_{i+1}} g^{-1}|$$
- (iii) $k_{n_{i+1}} < M_{n_i} \cdots M_{n_{i+1}-1}(i,k)$, *for every* $1 \leq i \leq k_{n_i}$.

Proof. We define $n_0 = 0$. Let $i \geq 0$ and suppose that we have defined n_j for every $0 \leq j \leq i$. Let $m_0 > n_i$ be such that for every $m \geq m_0$,

$$R_{n_i} \subseteq F_m.$$

Let $0 < \varepsilon < 1$ be such that $\varepsilon |R_{n_i}| < 1$. Since $(F_n)_{n\geq 0}$ is a Følner sequence, there exists $m_1 > m_0$ such that for every $m \geq m_1$,

$$\frac{|F_n \setminus F_m g^{-1}|}{|F_m|} < \frac{\varepsilon}{|F_{n_i+1}|}, \text{ for every } g \in R_{n_i}. \tag{8}$$

Since $\varepsilon |R_{n_i}| < 1$, there exists $m_2 > m_1$ such that for every $m \geq m_2$,

$$1 - |S_{n_i}| \frac{|F_{n_i+1}|}{|F_m|} > \varepsilon |R_{n_i}|.$$

Then

$$\frac{|F_m|}{|F_{n_i+1}|} - |S_{n_i}| > \varepsilon |R_{n_i}| \frac{|F_m|}{|F_{n_i+1}|}.$$

Conditions (1) and (3) for manageable sequences imply that

$$M_{n_i} \cdots M_{m-1}(l,j) \geq \frac{|F_m|}{|F_{n_i+1}|}, \text{ for every } 1 \leq l \leq k_{n_i}, 1 \leq j \leq k_m.$$

Combining the last two equations we get

$$M_{n_i} \cdots M_{m-1}(l,j) - |S_{n_i}| > \varepsilon |R_{n_i}| \frac{|F_m|}{|F_{n_i+1}|},$$

and from equation (8), we obtain

$$M_{n_i} \cdots M_{m-1}(l,j) - |S_{n_i}| > |F_m \setminus F_m g^{-1}||R_{n_i}|, \text{ for every } g \in R_{n_i},$$

which finally implies that

$$M_{n_i} \cdots M_{m-1}(l,j) > |\bigcup_{g\in R_{n_i}} F_m \setminus F_m g^{-1}| + |S_{n_i}|, \text{ for every } 1 \leq l \leq k_{n_i}, 1 \leq j \leq k_m.$$

Condition (4) for manageable sequences implies the existence of $m_3 > m_2$ such that for every $m \geq m_3$

$$k_{m+1} \leq M_{n_i} \cdots M_m(i,j) \text{ for every } 1 \leq i \leq k_n \text{ and } 1 \leq j \leq k_{n+1}.$$

By taking $n_{i+1} \geq m_3$ we get the desired subsequence $(n_i)_{i\geq 0} \subseteq \mathbb{Z}^+$. □

The following proposition shows that given a manageable sequence, there exists a sequence of decorations verifying conditions (C1)-(C4). The aperiodicity condition (C3) is obtained by decorating the center of F_n in a unique way with respect to other places in F_n. The restriction on the number of columns of the matrices in a manageable sequence, gives enough choices of coloring to ensure conditions (C3) and (C4).

Proposition 1. *Let* $(M_n)_{n\geq 0}$ *be a sequence of matrices which is manageable with respect to* $(|F_n|)_{n\geq 0}$. *For every* $n \geq 0$, *we denote by* k_n *the number of rows of* M_n. *Then there exists a Toeplitz subshift* $(X, \sigma|_X, G)$ *verifying the following three conditions:*

1. *The set of invariant probability measures of* $(X, \sigma|_X, G)$ *is affine homeomorphic to* $\varprojlim_n(\triangle(k_n, |F_n|), M_n)$.
2. *The ordered group* $\mathcal{G}(X, \sigma|_X, G)$ *is isomorphic to* $(H/inf(H), (H/inf(H))^+, u+inf(H))$, *where* $H = \varinjlim_n(\mathbb{Z}^{k_n}, M_n^T)$, $H^+ = \{[v, n] : v \geq 0, n \geq 0\}$ *and* $u = [|F_0|(1, \cdots, 1), 0]$.
3. $(X, \sigma|_X, G)$ *is an almost 1-1 extension of the odometer* $O = \varprojlim_n(G/\Gamma_n, \pi_n)$.

Proof. Let $(n_i)_{i\geq 0} \subseteq \mathbb{Z}^+$ be a sequence as in Lemma 11. Since $(M_n)_{n\geq 0}$ and the sequence $(M_{n_i} \cdots M_{n_{i+1}-1})_{i\geq 0}$ define the same inverse and direct limits, without loss of generality we can assume that for every $n \geq 0$ we have:

$$R_n \subseteq F_{n+1}, \tag{9}$$

$$M_n(i, k) > |S_n| + |\bigcup_{g\in R_n} F_{n+1} \setminus F_{n+1}g^{-1}| \text{ for every } 1 \leq i \leq k_n, 1 \leq k \leq k_{n+1}, \tag{10}$$

and

$$k_{n+1} < \min\{M_n(i, j) : 1 \leq i \leq k_n, 1 \leq j \leq k_{n+1}\}. \tag{11}$$

Claim 1. Let $C_{n,1}, \cdots, C_{n,r_n}$ be the connected components of $(F_{n+1} \cap \Gamma_n) \setminus (S_n \cup \{e\} \cup \bigcup_{g\in R_n} F_{n+1} \setminus F_{n+1}g^{-1})$ with respect to S_n. The number $M_n(1, k) - 1$ is smaller than the number of elements which are in all the components $C_{n,i}$ having more than one element.

To show the claim, notice that we have

$$|C_{n,1}| + \cdots + |C_{n,r_n}| \geq M_n(1, k) - 1 + M_n(2, k), \tag{12}$$

because

$$\begin{aligned}
M_n(1, k) + M_n(2, k) - 1 &\leq M_n(1, k) + \cdots + M_n(k_n - 1, k) - 1\\
&= |F_{n+1} \cap \Gamma_n| - M_n(k_n, 1) - 1\\
&\leq |F_{n+1} \cap \Gamma_n| - |S_n| - |\bigcup_{g\in R_n} F_{n+1} \setminus F_{n+1}g^{-1}| - |\{e\}|\\
&\leq |F_{n+1} \cap \Gamma_n \setminus (S_n \cup \bigcup_{g\in R_n} F_{n+1} \setminus F_{n+1}g^{-1} \cup \{e\})|\\
&= |C_{n,1}| + \cdots + |C_{n,r_n}|.
\end{aligned}$$

The claim is then trivial if for every $1 \leq i \leq r_n$ we have $|C_{n,i}| \geq 2$.
Suppose now, there is at least one component having only one element. Up to change the indexation, we may assume these components are $C_{n,1}, \cdots, C_{n,l_n}$ with $1 \leq l_n \leq r_n$. If $g \in C_{n,i}$ for $1 \leq i \leq l_n$, then for every $s \in S_n$ we have $gs \in F_{n+1}$ (because $g \notin \bigcup_{h\in R_n} F_{n+1} \setminus F_{n+1}h^{-1}$ and $S_n \subseteq R_n$), which implies that $gs \in S_n \cup \bigcup_{h\in R_n} F_{n+1} \setminus F_{n+1}h^{-1}$ ($gs \neq e$ because $g \notin S_n$). Thus if $s \in S_n$ and $g_i \in C_{n,i}$ for every $1 \leq i \leq l_n$, then $\{g_1s, \cdots, g_{l_n}s\} \subseteq S_n \cup \bigcup_{g\in R_n} F_{n+1} \setminus F_{n+1}g^{-1}$. Since this set has exactly l_n elements, we deduce

$$|S_n| + |\bigcup_{g\in R_n} F_{n+1} \setminus F_{n+1}g^{-1}| \geq |C_{n,1}| + \cdots + |C_{n,l_n}|.$$

Thus if $M_n(1,k)-1 > |C_{n,l_n+1}|+\cdots+|C_{n,r_n}|$, from (12) we get

$$\begin{aligned}|S_n| + |\bigcup_{g\in R_n} F_{n+1}\setminus F_{n+1}g^{-1}| &\geq |C_{n,1}|+\cdots+|C_{n,l_n}| \\ &\geq |C_{n,1}|+\cdots+|C_{n,r_n}| - (M_n(1,k)-1) \\ &\geq M_n(2,k),\end{aligned}$$

which contradicts (10) and shows the claim.

For every $n \geq 0$, we will construct a collection of functions $B_{n,1},\cdots,B_{n,k_n} \in \Sigma^{F_n}$ as in Lemma 10, where Σ is an alphabet such that $|\Sigma| = m \geq 3$. Namely $\Sigma = \{0,\cdots,m-1\}$.

Let $\mathcal{F}_0$ be the set of $B \in \Sigma^{F_0}$ verifying $B(e) = 0$ and $B(g) \in \Sigma\setminus\{0\}$ for every $g \in F_0\setminus\{e\}$. Since $|\mathcal{F}_0| \leq 2^{|F_0|-1}$ and since $k_0 \leq |F_0| \leq 2^{|F_0|-1}$, we can take $B_{0,1},\cdots,B_{0,k_0}$ different functions in $\mathcal{F}_0$. By definition, the collection $\{B_{0,1},\cdots,B_{0,k_0}\}$ verifies condition (C3).

Let $n \geq 0$. Suppose that we have defined $B_{n,1},\cdots,B_{n,k_n} \in \Sigma^{F_n}$ verifying condition (C3). For $1 \leq k \leq k_{n+1}$, we define

$$\sigma^{s^{-1}}(B_{n+1,k})|_{F_n} = B_{n,k_n} \text{ for every } s \in S_n \cup \bigcup_{g\in R_n} F_{n+1}\setminus F_{n+1}g^{-1}.$$

Now we will determine which are the rest of $v \in F_{n+1}\cap\Gamma_n$ for which $\sigma^{v^{-1}}(B_{n+1,k})|_{F_n} = B_{n,1}$. We set

$$B_{n+1,k}|_{F_n} = B_{n,1}.$$

Observe that equation (10) implies that $M_n(1,k) > 1$.

By Claim 1, we can write

$$M_n(1,k)-1 = \begin{cases} \sum_{i=l_n+1}^{p} |C_{n,i}| + r \text{ where } l_n+1 \leq p < r_n \text{ and } 1 \leq r \leq |C_{n,p+1}| & \text{case (1)} \\ \text{or} & \\ M_n(1,k)-1 = r \text{ where } 1 \leq r \leq |C_{n,l_n+1}| & \text{case (2),} \end{cases}$$

In the case (1) we define $\sigma^{g^{-1}}(B_{n+1,k})|_{F_n} = B_{n,1}$ for every $g \in C_{n,l_n+1}\cup\cdots\cup C_{n,p}$. Then, in both cases, if $r > 1$ we choose a connected set $D \subseteq C_{n,p+1}$ (for the first case) or $D \subseteq C_{n,l_n+1}$ (for the second case) such that $|D| = r$ (Lemma 9 ensure the existence of such set). Next, we define

$$\sigma^{v^{-1}}(B_{n+1,k})|_{F_n} = B_{n,1} \text{ for every } v \in D.$$

If $r = 1$, we choose a connected set $D \subseteq C_{n,p+1}$ (for the first case) or $D \subseteq C_{n,l_n+1}$ (for the second case) such that $|D| = 2$, namely $D = \{g, gs\}$, where $s \in S_n$. Then we define

$$\sigma^{g^{-1}}(B_{n+1,k})|_{F_n} = B_{n,1} \text{ and } \sigma^{(gs)^{-1}}(B_{n+1,k})|_{F_n} = B_{n,2}.$$

We fill the rest of the coordinates $v \in F_{n+1}\cap\Gamma_n$ in order that $\sigma^{v^{-1}}(B_{n+1,k})|_{F_n} \in \{B_{n,1},\cdots,B_{n,k_n}\}$ and such that

$$|\{v \in F_{n+1}\cap\Gamma_n : \sigma^{v^{-1}}(B_{n+1,k})|_{F_n} = B_{n,i}\}| = M_n(i,k),$$

for every $2 \leq i \leq k_n$. Notice that the number of such v is at least $M_n(2,k)$, because there are at least $M_n(2,k)-1$ coordinates to be filled with $B_{n,2}$ and at least 1 coordinate to be filled with B_{n,k_n}. Thus we have at least $M_n(2,k) > k_{n+1}$ different ways to fill the coordinates such that the functions $B_{n+1,1},\cdots,B_{n+1,k_{n+1}}$ are pairwise different.

By construction, every function $B_{n+1,k}$ verifies (C1), (C2) and (C4). Let us assume there are $g \in F_{n+1}$ and $1 \leq k,k' \leq k_{n+1}$ such that $B_{n+1,k}(gv) = B_{n+1,k'}(v)$ for any v where it is defined, then by the induction hypothesis, $g \in \Gamma_n$. This implies that $\sigma^{g^{-1}}(B_{n+1,k})|_{F_n} = B_{n+1,k'}|_{F_n} = B_{n,1}$. By definition, if $g \neq e$ then there exists $s \in S_n$ such that $gs \in F_{n+1}$

and $\sigma^{(gs)^{-1}}(B_{n+1,k})|_{F_n} = B_{n,1}$ or $B_{n,2}$. On the other hand, $\sigma^{s^{-1}}(B_{n+1,k'})|_{F_n} = B_{n,k_n}$. Since $\sigma^{(gs)^{-1}}(B_{n+1,k})|_{F_n}$ and $\sigma^{s^{-1}}(B_{n+1,k'})|_{F_n}$ have to coincide, we deduce that $g = e$. This shows that the collection $B_{n+1,1}, \cdots, B_{n+1,k_{n+1}}$ verifies (C3). We conclude applying Lemma 10. □

Proof of Theorem A. Let $ext(K)$ be the set of extreme points of K. If $ext(K)$ is finite, then the proof is direct from Lemma 2 and Proposition 1. If $ext(K)$ is infinite, the proof follows from Lemma 5 and Proposition 1. □

6. Proof of Theorem B.

In this section we briefly recall the concept of ordered Bratteli diagram and its relation with minimal $\mathbb{Z}$-actions on the Cantor set. We refer to [9] and [18] for the details.

A *Bratteli diagram* is an infinite directed graph $B = (V, E)$, such that the vertex set V and the edge set E can be partitioned into finite sets

$$V = V_0 \cup V_1 \cup \cdots \text{ and } E = E_1 \cup E_2 \cup \cdots$$

with the following properties:

- $V_0 = \{v_0\}$ is a singleton.
- For every $j \geq 1$, each edge in E_j starts from a vertex in V_{j-1} and arrives to a vertex in V_j.
- All vertices in V have at least one edge starting from it, and all vertices except v_0 have at least one edge arriving to it.

The sequence of *transition matrices* or *incidences matrices* of the Bratteli diagram B is the sequence $(M_n)_{n\geq 0}$ such that for every $n \geq 0$, the matrix $M_n \in \mathcal{M}_{V_n\times V_{n+1}}(\mathbb{Z}^+)$ is defined as

$$M_n(v, v') = \text{ number of edges from } v \in V_n \text{ to } v' \in V_{n+1}.$$

For a vertex $e \in E$ we denote by $s(e)$ the vertex where e starts and by $r(e)$ the vertex to which e arrives. For $m > n$, a *path* from $v \in V_n$ to $w \in V_m$ in B, is a sequence of edges $e_{n+1}e_{n+2}\cdots e_m$ such that $s(e_{n+1}) = v$, $r(e_m) = w$ and for each $n+1 \leq i \leq m-1$, $r(e_i) = s(e_{i+1})$. The set of *infinite paths* of B is

$$X_B = \{(e_i)_{i\geq 1} \in \prod_{i\geq 1} E_i : r(e_i) = s(e_{i+1}), \text{ for each } i \geq 1\}.$$

This set becomes a totally disconnected space when we endow every E_i with the discrete topology, $\prod_{i\geq 1} E_i$ with the product topology and X_B with the induced topology.

Let $(m_n)_{n\geq 0}$ be an increasing sequence of integers such that $m_0 = 0$. The *telescoping* of B to $(m_n)_{n\geq 0}$ is the Bratteli diagram $B' = (V', E')$ defined by $V' = \{V_{m_n} : n \geq 0\}$ and $E' = \{E'_n : n \geq 1\}$, where E'_n contains an edge from $v \in V_{m_{n-1}}$ to $w \in V_{m_n}$ in B' for each path from v to w in B. The Bratteli diagram B is *simple*, if there exists a telescoping B' of B so that the incidences matrices of B' are strictly positive. In this case, X_B is a Cantor set (when X_B is infinite).

An *ordered Bratteli diagram* $(V, E, \geq)$ is a Bratteli diagram $B = (V, E)$ together with a partial order $\geq$ on E, so that two edges are comparable if and only if they arrive at the same vertex. This order induces a transformation $T_B : X_B \to X_B$. If B is simple, the order $\geq$ can be chosen so that T_B is a minimal homeomorphism on X_B. Conversely, if $(X, T, \mathbb{Z})$ is a minimal Cantor system, then there exist a simple Bratteli diagram $B = (V, E)$ and an order $\geq$, such that $(V, E, \geq)$ is an ordered Bratteli diagram and $(X, T, \mathbb{Z})$ is conjugate to $(X_B, T_B, \mathbb{Z})$ (see [9] and [18] for the details). In this case we say that $B = (V, E)$ is an *associated Bratteli diagram* to $(X, T, \mathbb{Z})$. In [15], Giordano, Putnam and Skau show that the associated Bratteli diagram (without order) of a minimal $\mathbb{Z}$-action on the Cantor set is a total invariant of its strong orbit class.

We say that the Bratteli diagram $B = (V, E)$ has the *equal path number property* (e.p.n-property) if for each $n \geq 1$, $|r^{-1}(v)| = |r^{-1}(u)|$ for every $u, v \in V_n$. It is straightforward to show that the equal path number property is invariant under telescoping.

Lemma 12. *Every strong orbit equivalence class having an associated Bratteli diagram verifying the equal path number property, has an associated Bratteli diagram verifying the equal path number property and whose incidence matrices have at least three columns.*

Proof. Let $B = (V, E)$ be a Bratteli diagram verifying the e.p.n-property. If there exist infinitely many n's such that $|V_n| \geq 3$, we may telescope the diagram to those levels in order to get a new diagram B' whose incidence matrices have at least 3 columns. This new Bratteli diagram verifies the e.q.n-property and defines the same strong orbit equivalence class than B.
Otherwise, we can telescope the diagram B in order to get a new diagram B' whose incidence matrices have 1 or either 2 columns. In both cases, B' defines the same strong orbit equivalence class than a Bratteli diagram verifying the e.p.n-property and whose incidence matrices have exactly three columns. Indeed, let $(A_n)_{n\geq 0}$ be the sequence of incidence matrices of B' and suppose first that $A_n = [m_n]$, for every $n \geq 0$. We can assume that $m_n > 1$ (otherwise $X_{B'}$ contains only one point). Since $[m_n] = (1,1)(m_n - 1, 1)^T$, the diagram B' determines the same strong orbit equivalence class than the Bratteli diagram B'' defined by the sequence of incidence matrices $(B_n)_{n\geq 1}$, where $B_0 = (1,1)$ and $B_n = (m_{n-1} - 1, 1)^T(1,1)$ for every $n \geq 1$. The Bratteli diagram B'' determines the same orbit equivalence class than B', verifies the e.p.n-property and its incidences matrices have exactly two columns. Now we are in the case where every A_n has exactly two columns. Then we set $M_0 = A_0 A$ and $M_n = B_n A$ for $n \geq 1$, where

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } B_n = \begin{pmatrix} A_n(1,1) - 1 & A_n(1,2) - 1 \\ 1 & 1 \\ A_n(2,1) & A_n(2,2) \end{pmatrix}.$$

The Bratteli diagram B'' defined by the new sequence of matrices $(M_n)_{n\geq 0}$ determines the same strong orbit equivalence class than B', verifies the e.p.n-property and its incidence matrices have exactly 3 columns. ☐

The following proposition is the essential part of Theorem B.

Proposition 2. *Let $(X, \sigma|_X, \mathbb{Z})$ be a minimal Cantor system having an associated Bratteli diagram $B = (V, E)$ which satisfies the equal path number property. If in addition, there exists a constant $C > 0$ such that for each $n \geq 1$, $C|r^{-1}(v)| \geq |V_n|$ for every $v \in V_n$, then for every $d \geq 1$ there exists a Toeplitz subshift $(Y, \sigma|_Y, \mathbb{Z}^d)$ which is orbit equivalent to $(X, \sigma|_X, \mathbb{Z})$.*

In [16], Gjerde and Johansen show that every Toeplitz subshift $(X, \sigma|_X, \mathbb{Z})$ has an associated Bratteli diagram verifying the e.p.n-property. In [22], Sugisaki shows that every minimal Cantor system $(Y, T, \mathbb{Z})$ having an associated Bratteli diagram which satisfies the e.p.n-property, is strong orbit equivalent to a Toeplitz subshift $(X, \sigma|_X, \mathbb{Z})$. At the opposite of residually finite group, let us mention [11], where Dahl shows that any Brattteli diagram verifying the e.p.n-property is orbit equivalent to a free action of a locally finite group.

Proof of Proposition 2. From Lemma 12, we can assume that $|V_n| \geq 3$, for every $n \geq 1$.
Let $(M_n)_{n\geq 0}$ be the sequence of incidence matrices of B. For each $n \geq 1$, let $r_{n-1} = |r^{-1}(v)|$ for every $v \in V_n$. The e.p.n-property ensures

$$\sum_{v \in V_n} M_n(v, v') = r_n \text{ for every } v' \in V_{n+1} \text{ and } n \geq 0.$$

Let $(n_i)_{i\geq 0}$ be a sequence of non-negative integers such that $n_0=0$, $n_{i+1}>n_i+3$ and verifies $r_{n_i+1}\cdots r_{n_{i+1}-2}>C$, for every $i\geq 0$.
Observe that

$$M_{n_i}\cdots M_{n_{i+1}-1}(l,j)\geq r_{n_i+1}\cdots r_{n_{i+1}-1}, \text{ for every } l,j \text{ and } i\geq 0.$$

Defining $A_i=M_{n_i}\cdots M_{n_{i+1}-1}$, we have

$$A_i(l,j)>Cr_{n_{i+1}-1}\geq |V_{n_{i+1}}| \text{ for every } l,j \text{ and } i\geq 0.$$

The sequence $(A_i)_{i\geq 0}$ is the sequence of incidence matrices of the telescoping of B to the levels $(n_i)_{i\geq 0}$. This sequence is manageable with respect to $(p_i)_{i\geq 0}$, where $p_0=1$, $p_1=r_0\cdots r_{n_1-1}$ and $p_{i+1}=p_ir_{n_i}\cdots r_{n_{i+1}-1}$ for every $i\geq 1$. Let $u_{i+1}=r_{n_i}\cdots r_{n_{i+1}-1}$, for every $i\geq 0$.

Let $d\geq 1$ be an integer. For every $i\geq 1$ we define

$$q_i=p_0\prod_{j=1}^{id}u_j,$$

$$q_{i,1}=p_0\prod_{l=0}^{i-1}u_{dl+1} \text{ and } q_{i,j}=\prod_{l=0}^{i-1}u_{ld+j} \text{ for every } 2\leq j\leq d.$$

We have $p_{id}=q_i=q_{i,1}\cdots q_{i,d}$, for every $i\geq 1$.
Let $\Gamma_i=\prod_{j=1}^{d}q_{i,j}\mathbb{Z}$. The sequence of groups $(\Gamma_i)_{i\geq 1}$ satisfies $\Gamma_i\subseteq\Gamma_{i+1}$, $|\mathbb{Z}^d/\Gamma_i|=q_i$ and $\bigcap_{i\geq 1}\Gamma_i=\{0\}$. Thus from Proposition 1, there exists a Toeplitz subshift $(Y,\sigma|_Y,\mathbb{Z}^d)$ such that $\mathcal{G}(Y,\sigma|_Y,\mathbb{Z}^d)$ is isomorphic to $\mathcal{G}(X,\sigma|_X,\mathbb{Z})$. Theorem 2.5 in [14] implies that $(Y,\sigma|_Y,\mathbb{Z}^d)$ is orbit equivalent to $(X,\sigma|_X,\mathbb{Z})$. □

Corollary 1. *Let $(X,T,\mathbb{Z})$ be a minimal Cantor system whose dimension group is divisible. Then for every $d\geq 1$ there exists a Toeplitz subshift $(Y,\sigma|_Y,\mathbb{Z}^d)$ which is orbit equivalent to $(X,T,\mathbb{Z})$.*

Proof. From the proof of Theorem 12 in [16], there exists a Toeplitz subshift $(X',\sigma|_{X'},\mathbb{Z})$ which is orbit equivalent to $(X,T,\mathbb{Z})$, and such that $(X',\sigma|_{X'},\mathbb{Z})$ has an associated Bratteli diagram $B=(V,E)$ having the e.p.n-property and such that for each $n\geq 0$, $6|V_{n+1}|<r^{-1}(v)$ for every $v\in V_{n+1}$. We conclude applying Proposition 2. □

We combine Proposition 2 and Corollary 1 to obtain Theorem B.

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Departamento de Matemática y Ciencia de la Computación, Universidad de Santiago de Chile, Av. Libertador Bernardo O'Higgins 3363, Santiago, Chile. Laboratoire Amiénois de Mathématique Fondamentale et appliquée, CNRS U.M.R. 6140, Université de Picardie Jules Verne, 33 rue Saint-Leu 8039 Amiens Cedex 1 France.
E-mail address: `maria.cortez@usach.cl, samuel.petite@u-picardie.fr`