# Variational form of the large deviation functional

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### Abstract

Under exponential tightness hypothesis, we show that the large deviation functional $\overline{\Lambda}(\cdot)$ has the same variational form as when large deviations hold. As an application, we identify the affine regularization of a rate function on any real Hausdorff topological vector space in terms of $\overline{\Lambda}(\cdot)$. Some classical results involving the generalized log-moment generating function are improved.

**Keywords:** Large deviations; generalized log-moment generating function.

## 1 Introduction

Let $(\mu_\alpha)$ be a net of Borel probability measures on a topological space $X$, and let $(t_\alpha)$ be a net in $]0,+\infty[$ converging to 0. For any $[-\infty,+\infty[$-valued Borel measurable function $h$ on $X$, we write $\mu_\alpha^{t_\alpha}(e^{h/t_\alpha})$ for $(\int_X e^{h(x)/t_\alpha}\mu_\alpha(dx))^{t_\alpha}$, and define $\overline{\Lambda}(h) = \log\limsup \mu_\alpha^{t_\alpha}(e^{h/t_\alpha})$, and $\Lambda(h) = \log\lim \mu_\alpha^{t_\alpha}(e^{h/t_\alpha})$ when this limit exists. We denote by $\mathcal{C}(X)$ the set of $[-\infty,+\infty[$-valued continuous functions on $X$.

The usual version of Varadhan's theorem for $X$ Polish asserts that under exponential tightness hypothesis, a large deviation principle for $(\mu_\alpha^{t_\alpha})$ with rate function $J$ implies the existence of $\Lambda(h)$ for all $h \in \mathcal{C}(X)$ satisfying some tail condition, with

$$\Lambda(h) = \sup_{x\in X}\{h(x) - J(x)\}. \tag{1}$$

In this note, we show that if the large deviation hypothesis is dropped, then (1) remains true with $\overline{\Lambda}(h)$ in place of $\Lambda(h)$, and $J$ replaced by the function $l_0$, where $l_0(x) = -\log\inf\{\limsup \mu_\alpha^{t_\alpha}(G) : x \in G \subset X, G \text{ open}\}$ for all $x \in X$ (note that a large deviation principle on a regular space is always governed by $l_0$); furthermore, this is valid for any topological space in which compact sets are Borel sets (Theorem 1). If moreover the exponential tightness is dropped, then the result remains true replacing $\mathcal{C}(X)$ by the set $\mathcal{C}_{\mathcal{K}}(X)$ of elements $h \in \mathcal{C}(X)$ for which $\{y \in X : e^{h(x)} - \varepsilon \leq e^{h(y)} \leq e^{h(x)} + \varepsilon\}$ is compact for all $x \in X$ and $\varepsilon > 0$ with $e^{h(x)} > \varepsilon$. A general version of Varadhan's theorem is a direct consequence (Corollary 1).

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In the last section we apply Theorem 1 to the case where $X$ is a real Hausdorff topological vector space and $h$ belongs to its topological dual $X^*$. Assuming the exponential tightness and large deviations for $(\mu_\alpha^{t_\alpha})$ with rate function $J$, we show that the affine regularization of $J$ is the Legendre-Fenchel transform $\tilde{L}^*$ of the map $\tilde{L}$ defined on $X^*$ by $\tilde{L}(\lambda) = \lim_{M\to+\infty} \overline{\Lambda}(\lambda 1_{\{\lambda\leq M\}} - \infty 1_{\{\lambda > M\}})$. It follows that $J = \tilde{L}^*$ when $J$ is convex and $X$ locally convex. We show that some classical results (and in particular the equality $J = \overline{L}^*$ or even $J = L^*$) known when $\overline{L} < \infty$ are still valid under weaker conditions (Corollaries 2, 3); in particular, Baldi's theorem is slightly improved (Theorem 2).

### 1.1 Notations

We recall that $(\mu_\alpha)$ satisfies a large deviation principle with powers $(t_\alpha)$ if there exists a $[0, +\infty]$-valued lower semi-continuous function $J$ on $X$ such that

$$\limsup \mu_\alpha^{t_\alpha}(F) \leq \sup_{x\in F} e^{-J(x)} \qquad \text{for all closed } F \subset X, \tag{2}$$

and

$$\sup_{x\in G} e^{-J(x)} \leq \liminf \mu_\alpha^{t_\alpha}(G) \qquad \text{for all open } G \subset X;$$

$J$ is called a rate function for $(\mu_\alpha^{t_\alpha})$, which is said to be tight when it has compact level sets. When "closed" is replaced by "compact" in (2), we say that a vague large deviation principle holds. The net $(\mu_\alpha)$ is exponentially tight with respect to $(t_\alpha)$ if for each $\varepsilon > 0$ there exists a compact $K \subset X$ such that $\limsup \mu_\alpha^{t_\alpha}(X\backslash K) < \varepsilon$. For each $[-\infty, +\infty[$-valued Borel measurable function $h$ on $X$, each $a \geq 0$ and each $\varepsilon > 0$, we set $G_{a,\varepsilon}^h = \{x \in X : a - \varepsilon < e^{h(x)} < a + \varepsilon\}$ and $F_{a,\varepsilon}^h = \{x \in X : a - \varepsilon \leq e^{h(x)} \leq a + \varepsilon\}$ (we simply write $G_{a,\varepsilon}$, $F_{a,\varepsilon}$ when $a = e^{h(z)}$ for some $z \in X$).

When $X$ is a real Hausdorff topological vector space, we denote by $X^*$ its topological dual endowed with the weak$^*$-topology. For each map $f : X \mapsto [-\infty, +\infty]$, we define a map $f^*$ on $X^*$ by $f^*(\lambda) = \sup_{x\in X}\{\lambda(x) - f(x)\}$ for all $\lambda \in X^*$. By applying this definition to $f^*$, one obtains a map $f^{**}$ on $X$ defined by $f^{**}(x) = \sup_{\lambda\in X^*}\{\lambda(x) - f^*(\lambda)\}$ for all $x \in X$. It is easy to see that $f^{**}$ coincides with the affine-regularization of $f$ (i.e., the pointwise supremum of the continuous affine functions everywhere less than $f$). When moreover $X$ is locally convex and $f$ is convex lower semi-continuous with the property that $f \equiv -\infty$ when $f$ takes the value $-\infty$, then $f^{**} = f$ (Ekeland and Teman, 1976, Proposition 4.1).

## 2 Variational representation for $\overline{\Lambda}(\cdot)$

Under exponential tightness hypothesis, part $(b)$ of the following theorem gives a necessary and sufficient condition in order that $\overline{\Lambda}(h)$ have the same form as when large deviations hold (see Comman, 2003, for a general version of Varadhan's theorem valid on any topological space and without tightness hypothesis). This condition is strictly weaker than the usual Varadhan's tail condition, since this one requires that the L.H.S. of (5) vanishes.

**Theorem 1** *Let $X$ be a topological space in which compact sets are Borel sets, let $h \in \mathcal{C}(X)$, and assume that either $h \in \mathcal{C}_\mathcal{K}(X)$ or $(\mu_\alpha)$ is exponentially tight with respect to $(t_\alpha)$. The following conclusions hold:*

*(a) For each real $M$, we have*

$$\sup_{x\in\{h<M\}} e^{h(x)-l_0(x)} \leq \limsup \mu_\alpha^{t_\alpha}(e^{h/t_\alpha}1_{\{h\leq M\}}) \leq \sup_{x\in\{h\leq M\}} e^{h(x)-l_0(x)}. \tag{3}$$

*In particular,*

$$\lim_{M\to+\infty} \limsup \mu_\alpha^{t_\alpha}(e^{h/t_\alpha}1_{\{h\leq M\}}) = \sup_{x\in X} e^{h(x)-l_0(x)}. \tag{4}$$

*(b) $\overline{\Lambda}(h) = \sup_{x\in X}\{h(x)-l_0(x)\}$ if and only if*

$$\lim_{M\to+\infty} \limsup \mu_\alpha^{t_\alpha}(e^{h/t_\alpha}1_{\{h>M\}}) \leq \lim_{M\to+\infty} \limsup \mu_\alpha^{t_\alpha}(e^{h/t_\alpha}1_{\{h\leq M\}}). \tag{5}$$

*If the above inequality is strict, then for each real $M$ large enough, we have*

$$\overline{\Lambda}(h) = \sup_{x\in\{h\leq M\}}\{h(x)-l_0(x)\}. \tag{6}$$

*Proof.* For each real $M$, we put $h_M = h1_{\{h\leq M\}} - \infty 1_{\{h>M\}}$ (with the convention "$\infty\cdot 0 = 0$"). By Theorem 3.1 of Comman (2003) applied to $h_M$, we get

$$\limsup \mu_\alpha^{t_\alpha}(e^{h_M/t_\alpha}1_{\{h\leq M\}}) =$$

$$\sup_{x\in\{h\leq M\},\varepsilon>0}\{(e^{h_M(x)}-\varepsilon)\limsup \mu_\alpha^{t_\alpha}(F_{e^{h_M(x)},\varepsilon}\cap\{h\leq M\})\}.$$

Since $h_M$ coincides with $h$ on $\{h\leq M\}$, we have $F_{e^{h_M(x)},\varepsilon}\cap\{h\leq M\} = F_{e^{h(x)},\varepsilon}\cap\{h\leq M\}$, hence

$$\limsup \mu_\alpha^{t_\alpha}(e^{h/t_\alpha}1_{\{h\leq M\}}) = \sup_{x\in\{h\leq M\},\varepsilon>0}\{(e^{h(x)}-\varepsilon)\limsup \mu_\alpha^{t_\alpha}(F_{e^{h(x)},\varepsilon}\cap\{h\leq M\})\}$$

$$\geq \sup_{x\in\{h<M\},\varepsilon>0}\{(e^{h(x)}-\varepsilon)\limsup \mu_\alpha^{t_\alpha}(G_{e^{h(x)},\varepsilon}\cap\{h<M\})\} \geq \sup_{x\in\{h<M\}} e^{h(x)-l_0(x)}, \tag{7}$$

which proves the first inequality of (3). Since the second inequality of (3) holds obviously when $\limsup \mu_\alpha^{t_\alpha}(e^{h/t_\alpha}1_{\{h\leq M\}}) = 0$, we assume that $\limsup \mu_\alpha^{t_\alpha}(e^{h}1_{\{h\leq M\}}) > e^m$ for some real $m$. Suppose that

$$\sup_{x\in\{h\leq M\}} e^{h(x)-l_0(x)} + \nu < \limsup \mu_\alpha^{t_\alpha}(e^{h/t_\alpha}1_{\{h\leq M\}})$$

for some $\nu > 0$. First assume that exponential tightness holds, and let $K\subset X$ be a compact set such that $\limsup \mu_\alpha^{t_\alpha}(X\backslash K) < e^{m-M}$, and get by (7),

$$\limsup \mu_\alpha^{t_\alpha}(e^{h/t_\alpha}1_{\{h\leq M\}}) = \sup_{x\in\{h\leq M\},\varepsilon>0}\{(e^{h(x)}-\varepsilon)\limsup \mu_\alpha^{t_\alpha}(F_{e^{h(x)},\varepsilon}\cap\{h\leq M\}\cap K)\}.$$

Then, there exists $x_0\in\{h\leq M\}$ and $\varepsilon_0 > 0$ with $e^{h(x_0)} > \varepsilon_0$ such that

$$\sup_{x\in\{h\leq M\}} e^{h(x)-l_0(x)} < (e^{h(x_0)}-\varepsilon_0-\nu)\limsup \mu_\alpha^{t_\alpha}(F_{e^{h(x_0)},\varepsilon_0}\cap\{h\leq M\}\cap K). \tag{8}$$

For each $x\in F_{e^{h(x_0)},\varepsilon_0}\cap\{h\leq M\}\cap K$, there is an open set $V_x$ containing $x$, and such that $e^{h(y)} > e^{h(x_0)}-\varepsilon_0-\nu$ for all $y\in V_x$. By (8), for each $x\in F_{e^{h(x_0)},\varepsilon_0}\cap\{h\leq M\}\cap K$, there is an open set $W_x$ containing $x$, and such that

$$e^{h(x)}\limsup \mu_\alpha^{t_\alpha}(W_x) < (e^{h(x_0)}-\varepsilon_0-\nu)\limsup \mu_\alpha^{t_\alpha}(F_{e^{h(x_0)},\varepsilon_0}\cap\{h\leq M\}\cap K). \tag{9}$$

Put $G_x = W_x \cap V_x$ for all $x \in F_{e^{h(x_0)},\varepsilon_0} \cap \{h \leq M\} \cap K$. Since $F_{e^{h(x_0)},\varepsilon_0} \cap \{h \leq M\} \cap K$ is compact, there is a finite set $A \subset F_{e^{h(x_0)},\varepsilon_0} \cap \{h \leq M\} \cap K$ such that $F_{e^{h(x_0)},\varepsilon_0} \cap \{h \leq M\} \cap K \subset \bigcup_{x \in A} G_x$, hence for some $x \in A$,

$$(e^{h(x_0)} - \varepsilon_0 - \nu) \limsup \mu_\alpha^{t_\alpha}(F_{e^{h(x_0)},\varepsilon_0} \cap \{h \leq M\} \cap K) \leq e^{h(x)} \limsup \mu_\alpha^{t_\alpha}(G_x), \quad (10)$$

which contradicts (9); therefore, $\limsup \mu_\alpha^{t_\alpha}(e^{h/t_\alpha} 1_{\{h \leq M\}}) \leq \sup_{x \in \{h \leq M\}} e^{h(x) - l_0(x)}$ and (3) holds. If $h \in \mathcal{C}_\mathcal{K}(X)$, then the above proof works verbatim with $F_{e^{h(x_0)},\varepsilon_0} \cap \{h \leq M\}$ in place of $F_{e^{h(x_0)},\varepsilon_0} \cap \{h \leq M\} \cap K$ in (8), (9), (10), since $F_{e^{h(x_0)},\varepsilon_0} \cap \{h \leq M\}$ is compact. Then, (4) is a direct consequence of (3), and so $(a)$ holds. The first assertion of $(b)$ follows from (4) and the fact that for each real $M$,

$$\limsup \mu_\alpha^{t_\alpha}(e^{h/t_\alpha}) = \limsup \mu_\alpha^{t_\alpha}(e^{h/t_\alpha} 1_{\{h \leq M\}}) \vee \limsup \mu_\alpha^{t_\alpha}(e^{h/t_\alpha} 1_{\{h > M\}}).$$

If the inequality in (5) is strict, then $\limsup \mu_\alpha^{t_\alpha}(e^{h/t_\alpha}) = \limsup \mu_\alpha^{t_\alpha}(e^{h/t_\alpha} 1_{\{h \leq M\}})$ for all $M$ large enough, and (6) follows from $(a)$. $\square$

The following corollary shows that under exponential tightness, and for a given $h \in C(X)$, all the conclusions of Varadhan's theorem remain true with $l_0$ in place of the rate function, and replacing the large deviation hypothesis by the weaker condition (11); the existence of $\Lambda(h)$ follows from Corollary 1 of Comman (2005) by noting that its conclusion still holds when the usual tail condition is replaced by a strict inequality in (5). Note that $F^h_{a,\delta}$ is compact when $h \in \mathcal{C}_\mathcal{K}(X)$ and $a - \delta > 0$, so that (11) is weaker than vague large deviations.

**Corollary 1** *Let $X$ be a topological space in which compact sets are Borel sets, let $h \in \mathcal{C}(X)$ satisfying (5) strictly and*

$$\limsup \mu_\alpha^{t_\alpha}(F^h_{a,\delta}) \leq \liminf \mu_\alpha^{t_\alpha}(G^h_{a,\varepsilon}) \quad (11)$$

*for all reals $a \wedge \varepsilon > \delta > 0$. If either $(\mu_\alpha)$ is exponentially tight with respect to $(t_\alpha)$ or $h \in \mathcal{C}_\mathcal{K}(X)$, then $\Lambda(h)$ exists and for each real $M$ large enough,*

$$\Lambda(h) = \sup_{x \in \{h \leq M\}} \{h(x) - l_0(x)\} = \sup_{x \in X} \{h(x) - l_0(x)\}.$$

# 3 The affine regularization of a rate function

In is section, $X$ is a real Hausdorff topological vector space, and $X^*$ denotes its topological dual. Let us consider the maps $\tilde{L}$ and $\overline{L}$ defined on $X^*$ respectively by

$$\tilde{L}(\lambda) = \lim_{M \to +\infty} \limsup t_\alpha \log \mu_\alpha(e^{\lambda/t_\alpha} 1_{\{\lambda \leq M\}})$$

and

$$\overline{L}(\lambda) = \limsup t_\alpha \log \mu_\alpha(e^{\lambda/t_\alpha})$$

for all $\lambda \in X^*$; we write $L(\lambda)$ when the limit exists in the last expression ($L$ is then the usual generalized log-moment generating function). Note that $\tilde{L}(0) = 0$, and $\tilde{L}$ is convex since the map $\lambda \mapsto \log \mu_\alpha(e^{\lambda/t_\alpha} 1_{\{h \leq M\}})$ is convex by Hölder's inequality. If $X = \mathbb{R}$ or $(\mu_\alpha)$ is exponentially tight with respect to $(t_\alpha)$, then $\tilde{L}$ is weak$^*$ lower semicontinuous by (4), which proves part (a) of Corollary 2. Part (c) follows from (a), (b) and the fact that the finiteness of $\overline{L}$ implies the Varadhan's tail condition for all $\lambda \in X^*$

(Dembo and Zeitouni, 1998, Lemma 4.3.8); (c) was known under the hypotheses that $X$ is locally convex and large deviations hold with tight rate function (Dembo and Zeitouni, 1998, Remark pp. 153).

**Corollary 2** *Let $X$ be a real Hausdorff topological vector space. If $X = \mathbb{R}$ or if $(\mu_\alpha)$ is exponentially tight with respect to $(t_\alpha)$, then*

*(a) $\tilde{L} = {l_0}^*$; in particular, $\tilde{L}$ is weak$^*$ lower semi-continuous.*

*(b) For each $\lambda \in X^*$, $\tilde{L}(\lambda) = \overline{L}(\lambda)$ if and only if (5) holds with $h = \lambda$.*

*(c) $\overline{L}$ is weak$^*$ lower semi-continuous when it is finite-valued.*

We assume now that $(\mu_\alpha)$ satisfies a large deviation principle with powers $(t_\alpha)$ and rate function $J$. When $\overline{L}$ does not take the value $+\infty$, each $\lambda \in X^*$ satisfies the tail condition of Varadhan's theorem, hence by the tightness-free version of this theorem (Comman, 2003, Corollary 3.4), $L(\lambda)$ exists for all $\lambda \in X^*$ and $L = J^*$, and we get the following result, which is usually stated in the literature under the extra hypothesis that $J$ is tight (see Deuschel and Strook, 1989, Theorem 2.2.21; Dembo and Zeitouni, 1998, Theorem 4.5.10):

- $L^*$ is the affine regularization of $J$, and $J$ is convex if and only if $J = L^*$ when moreover $X$ is locally convex.

When $\overline{L}$ takes the value $+\infty$, the Varadhan's theorem cannot be applied in order to get the existence of $L(\lambda)$. However, when exponential tightness holds we can use Corollary 1 to get a strictly weaker condition than the finiteness of $\overline{L}$ in order that the above result remains true; with a bit weaker condition it suffices to replace $L$ by $\overline{L}$; this is the content of part (b) of Corollary 3. Part (a) shows that in any case it holds verbatim replacing $L$ by $\tilde{L}$. Moreover, it suffices to assume vague large deviations when $X = \mathbb{R}$; in particular any convex rate function governing a vague large deviation principle is given by $\tilde{L}^*$.

**Corollary 3** *We assume that one of the following conditions holds:*

*(i) $X = \mathbb{R}$ and $(\mu_\alpha)$ satisfies a vague large deviation principle with powers $(t_\alpha)$ and rate function $J$.*

*(ii) $X$ is a real Hausdorff topological vector space, $(\mu_\alpha)$ is exponentially tight with respect to $(t_\alpha)$, and $(\mu_\alpha)$ satisfies a large deviation principle with powers $(t_\alpha)$ and rate function $J$.*

*Then,*

*(a) $\tilde{L}^*$ is the affine regularization of $J$. When moreover $X$ is locally convex, $J$ is convex if and only if $J = \tilde{L}^*$.*

*(b) If (5) holds (resp. holds strictly) for all $\lambda \in X^*$, then (a) holds verbatim with $\overline{L}$ (resp. $L$) in place of $\tilde{L}$.*

*Proof.* Since $J = l_0$ we get $\tilde{L} = J^*$ by Corollary 2 (*a*), which proves the first assertion of (*a*); the second one follows from local convexity. The first assertion of (*b*) follows from Corollary 2 (*b*); the existence of $L(\lambda)$ in the second assertion follows from Corollary 1. $\square$

**Example 1** Consider the net $(\mu_\varepsilon^\varepsilon)_{\varepsilon>0}$, where $\mu_\varepsilon$ is the probability measure on $\mathbb{R}$ defined by $\mu_\varepsilon(0) = 1 - 2p_\varepsilon$, $\mu_\varepsilon(-\varepsilon \log p_\varepsilon) = \mu_\varepsilon(\varepsilon \log p_\varepsilon) = p_\varepsilon$, and assume that $\lim \varepsilon \log p_\varepsilon = -\infty$. It is easy to see that $(\mu_\varepsilon^\varepsilon)_{\varepsilon>0}$ satisfies a large deviation principle with the convex rate function

$$J(x) = \begin{cases} 0 & \text{if } x = 0 \\ +\infty & \text{otherwise.} \end{cases}$$

For each real $\lambda$, we have

$$L(\lambda) = \lim \varepsilon(\lambda + 1) \log p_\varepsilon \vee \lim \varepsilon \log(1 - 2p_\varepsilon) \vee \lim \varepsilon(1 - \lambda) \log p_\varepsilon,$$

hence

$$L(\lambda) = \begin{cases} 0 & \text{if } |\lambda| \leq 1 \\ +\infty & \text{if } |\lambda| > 1, \end{cases}$$

and $L^*(x) = |x|$ for all $x \in \mathbb{R}$. On the other hand, we have

$$\tilde{L}(\lambda) = \begin{cases} \lim \varepsilon(\lambda + 1) \log p_\varepsilon \vee \lim \varepsilon \log(1 - 2p_\varepsilon) = 0 & \text{if } \lambda \geq 0 \\ \lim \varepsilon \log(1 - 2p_\varepsilon) \vee \lim \varepsilon(1 - \lambda) \log p_\varepsilon = 0 & \text{if } \lambda < 0, \end{cases}$$

and so $J = \tilde{L}^* \neq L^*$.

We give now a slight strengthening of Baldi's theorem by asking only that (5) holds strictly for the exposed hyperplanes $\lambda$ for which $L(\lambda)$ exists, in place of the stronger hypothesis $\overline{L}(c\lambda) < \infty$ for some $c > 1$ (which implies the Varadhan's tail condition for $\lambda$, and which is actually used in the usual proofs). Note that our proof is very short and direct in comparison.

**Theorem 2** *Let $X$ be a real Hausdorff topological vector space and assume that $(\mu_\alpha)$ is exponentially tight with respect to $(t_\alpha)$. Let $\mathcal{E}$ be the set of exposed points $x$ of $\overline{L}^*$ for which there is an exposed hyperplane $\lambda_x$ such that $L(\lambda_x)$ exists and (5) holds strictly. If $\inf_G \overline{L}^* = \inf_{G \cap \mathcal{E}} \overline{L}^*$ for all open $G \subset X$, then $(\mu_\alpha)$ satisfies a large deviation principle with powers $(t_\alpha)$ and rate function $\overline{L}^*$.*

*Proof.* Let $\delta > \sup_{x \in K} e^{-\overline{L}^*(x)}$ for some compact $K \subset X$ and some real $\delta$. For each $x \in K$ there exists $\lambda_x \in X^*$ such that

$$\delta > e^{\overline{L}(\lambda_x) - \lambda_x x} \geq e^{-\lambda_x x} \sup_{z \in X} e^{\lambda_x z - l_0(z)},$$

where the last inequality follows from (4), and by taking $z = x$ we get $\delta > e^{-l_0(x)}$ and so $\delta > \limsup \mu_\alpha^{t_\alpha}(G_x)$ for some open $G_x$ containing $x$. Since $K$ is compact it can be covered by a finite number of such $G_x$, which gives $\delta > \limsup \mu_\alpha^{t_\alpha}(K)$; this proves the upper bounds since $K$ and $\delta$ are arbitrary. Let $x \in \mathcal{E}$ with some exposed hyperplane $\lambda_x$ for which $L(\lambda_x)$ exists and (5) holds strictly. Let $(\mu_\beta^{t_\beta})$ be a subnet of $(\mu_\alpha^{t_\alpha})$. Since $X$ is regular, by exponential tightness $(\mu_\beta)$ has a subnet $(\mu_\gamma)$ satisfying a large deviation principle with powers $(t_\gamma)$ and rate function $l_0^{(\mu_\gamma^{t_\gamma})}$ (where $l_0^{(\mu_\gamma^{t_\gamma})}$ is the function obtained replacing $\mu_\alpha^{t_\alpha}$ by $\mu_\gamma^{t_\gamma}$ in the definition of $l_0$). By Theorem 1 applied to the net $(\mu_\gamma^{t_\gamma})$, and since $L(\lambda_x)$ exists, there is $x_0 \in X$ such that $L(\lambda_x) = \lambda_x x_0 - l_0^{(\mu_\gamma^{t_\gamma})}(x_0)$. We have

$$\lambda_x x_0 - \overline{L}^*(x_0) \geq \lambda_x x_0 - l_0(x_0) \geq \lambda_x x_0 - l_0^{(\mu_\gamma^{t_\gamma})}(x_0) = L(\lambda_x) \geq L^{**}(\lambda_x) = \lambda_x x - \overline{L}^*(x),$$

where the first inequality follows by noting that $\overline{L}^* \leq l_0$ since $l_0{}^* \leq \overline{L}$ by (4). Since $x$ is exposed with exposed hyperplane $\lambda_x$, all the above inequalities are equalities and $x = x_0$

hence $l_0^{(\mu_\gamma^{t_\gamma})}(x) = \overline{L}^*(x)$. Since large deviation hold for $(\mu_\gamma^{t_\gamma})$, we have $l_0^{(\mu_\gamma^{t_\gamma})} = l_1^{(\mu_\gamma^{t_\gamma})}$ (where $l_1^{(\mu_\gamma^{t_\gamma})}(x) = -\log\inf\{\liminf \mu_\gamma^{t_\gamma}(G) : x \in G \subset X, G \text{ open}\}$), so that $l_1^{(\mu_\gamma^{t_\gamma})}(x) = \overline{L}^*(x)$. Therefore, $\liminf \mu_\gamma^{t_\gamma}(G) \geq e^{-\overline{L}^*(x)}$ for all open $G$ containing $x$, and since $x$ is arbitrary in $\mathcal{E}$, we get

$$\liminf \mu_\gamma^{t_\gamma}(G) \geq \sup_{x \in G \cap \mathcal{E}} e^{-\overline{L}^*(x)} = \sup_{x \in G} e^{-\overline{L}^*(x)}.$$

Since the upper bounds hold obviously for $(\mu_\gamma^{t_\gamma})$, we have proved that any subnet of $(\mu_\alpha^{t_\alpha})$ has a subnet satisfying a large deviation principle with rate function $\overline{L}^*$. Seing a large deviation principle as a convergence in a suitable space of set-functions (see Comman, 2003, Remark 3.5), we get the same result for $(\mu_\alpha^{t_\alpha})$. □

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