UPPER REGULARIZATION FOR EXTENDED SELF-ADJOINT OPERATORS

HENRI COMMAN

Abstract. We show that the complete lattice of $\overline{\mathbb{R}}$-valued sup-preserving maps on a complete lattice $\mathcal{G}$ of projections of a von Neumann algebra $\mathcal{M}$, is isomorphic to some complete lattice $\mathcal{M}_{\overline{\mathbb{R}}}^{\mathcal{G}}$ of extended spectral families in $\mathcal{M}$, provided with the spectral order. We get various classes of (not necessarily densely defined) self-adjoint operators affiliated with $\mathcal{M}$ as conditionally complete lattices with completion $\mathcal{M}_{\overline{\mathbb{R}}}^{\mathcal{G}}$, extending the Olson's results. When $\mathcal{M}$ is the universal enveloping von Neumann algebra of a C^*-algebra A, and $\mathcal{G}$ the set of open projections, the elements of $\mathcal{M}_{\overline{\mathbb{R}}}^{\mathcal{G}}$ are said to be extended q-upper semicontinuous, generalizing the usual notions. The q-upper regularization map is defined using the spectral order, and characterized in terms of the above isomorphism. When A is commutative with spectrum X, we give an isomorphism Π of complete lattices from $\overline{\mathbb{R}}^X$ into the set of extended self-adjoint operators affiliated with $\mathcal{M}$. By means of Π, the above characterizations appear as generalizations of well-known properties of the upper regularization of $\overline{\mathbb{R}}$-valued functions on X. A noncommutative version of the Dini-Cartan's lemma is given. An application is sketched.

1. Introduction

The spectral order has been first considered by Olson ([15]), and next appears in various contexts ([3], [4], [13]). In noncommutative topology, and by means of spectral projections, L. G. Brown defines a notion of semicontinuity for self-adjoint operators in the universal enveloping von Neumann algebra A'' of a C^*-algebra A, the so-called q-semicontinuity; the usual notion for bounded functions on a locally compact Hausdorff space X is recovered by taking A commutative with spectrum X. Brown observes that in general, the q-lower semicontinuity is not preserved by strongly increasing nets ([6], pp. 905). However, since the q-semicontinuity is defined spectrally, it is natural to work with it by considering the spectral order, and particularly the fact that the self-adjoint part of a von Neumann algebra provided with this order is a conditionally complete lattice; as an immediate consequence, the above drawback disappears since the join for the spectral order of any bounded family of q-lower semicontinuous operators is q-lower semicontinuous. It is easy to see that for A completely σ-unital, the open projections can be equivalently defined via the spectral order in place of the usual one; moreover, each q-lower semicontinuous operator is the join of some set in the self-adjoint part of $\tilde{A}$, where $\tilde{A}$ is the unitization of A. So, the q-semicontinuity seems to behave particularly

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well with the spectral order. We then can hope to extend somewhat more involved properties of functions related to semicontinuity and order.

The aim of this paper is to extend to general C^*-algebras, the essential properties of the upper regularization map $f \mapsto \overline{f}$ defined on the set $\overline{\mathbb{R}}^X$ of $[-\infty, +\infty]$-valued functions on X, where $\overline{f}$ denotes the least upper semicontinuous function greater than f. In fact, these properties will be generalized even in the commutative case.

In order to handle the case where f takes infinite values, we introduce in Section 2 the notion of extended self-adjoint operator affiliated with a von Neumann algebra $\mathcal{M}$ (Definition 2.1). Let $\mathcal{P}$ denote the set of projections of $\mathcal{M}$, and $\mathcal{G} \subset \mathcal{P}$ be a complete lattice containing the identity operator, and in which the joins coincide with the joins in $\mathcal{P}$. The set $\mathcal{M}_{\overline{\mathbb{R}}}^{\mathcal{G}}$ of extended self-adjoint operators x satisfying $E^x_{]-\infty,\lambda[} \in \mathcal{G}$ for all reals λ, provided with the spectral order is then a complete lattice. More precisely, Theorem 2.3 gives an isomorphism $\psi^{\mathcal{G}}$ of partially ordered set between $\mathcal{M}_{\overline{\mathbb{R}}}^{\mathcal{G}}$ and the complete lattice $\mathrm{Sup}(\mathcal{G}, \overline{\mathbb{R}})$ of $\overline{\mathbb{R}}$-valued sup-preserving maps on $\mathcal{G}$, provided with the usual order on functions (we shall write simply ψ when $\mathcal{G} = \mathcal{P}$). As a direct consequence, $\mathcal{M}_{\overline{\mathbb{R}}}^{\mathcal{G}} \cap \mathcal{M}$ is a conditionally complete lattice (recovering the Olson's result with $\mathcal{G} = \mathcal{P}$) for which $\mathcal{M}_{\overline{\mathbb{R}}}^{\mathcal{G}}$ is a completion. Other conditionally complete lattices are obtained by requiring the extended spectrum to be in a given set (Corollary 2.4).

In Section 3, we take $\mathcal{M} = A''$ and $\mathcal{G}$ the set of open projections. The elements of $A''^{\mathcal{G}}_{\overline{\mathbb{R}}}$ are said to be extended q-upper semicontinuous, generalizing the usual notions. Since $A''^{\mathcal{G}}_{\overline{\mathbb{R}}}$ is a complete lattice, we can define the q-upper regularization $\overline{x}$ for any $x \in A''_{\overline{\mathbb{R}}}$ in a natural way (Definition 3.1). The map $x \mapsto \overline{x}$ is a closure operator which extends the one defined by Akemann on projections (Proposition 3.2); it is then characterized in terms of ψ and $\psi^{\mathcal{G}}$ in Theorem 3.3, and Theorem 3.4 looks like a Dini-Cartan's lemma.

In Section 4, we prove that the two above theorems are not only formal analogues of classical results for functions, but true generalizations. For this purpose, we study in detail the case where A is commutative with spectrum X. Theorem 4.2 gives an isomorphism of complete lattices Π from $\overline{\mathbb{R}}^X$ into $A''_{\overline{\mathbb{R}}}$ satisfying all the required properties, and in particular $\overline{\Pi(f)} = \Pi(\overline{f})$ for all $f \in \overline{\mathbb{R}}^X$. Then, taking A commutative in Theorem 3.3, the properties of the upper regularization map are recovered by restriction on the image of Π and using the above equality (Corollary 4.4). The Dini-Cartan's lemma is obtained as the commutative case of Theorem 3.4 (Corollary 4.5).

In Section 5, the properties of the q-upper regularization map are used to obtain a noncommutative version of a basic result in large deviation theory.

1.1. **Notations and background material.** Throughout the paper, A is a C^*-algebra considered as a C^*-subalgebra of its universal enveloping von Neumann algebra A''. A projection $p \in A''$ is open if there is an increasing net in the positive part of A converging strongly to p. A projection $p \in A''$ is closed if $1 - p$ is open, where 1 is the unit of A''. A closed projection p is compact if there is a positive $x \in A$ such that $p \leq x$. The join of any set of open projections is an open projection. Let x be a self-adjoint operator on some closed subspace of the universal Hilbert space, and affiliated with A''. Then, x is q-upper semicontinuous if $E^x_{]-\infty,\lambda[}$ is open for all reals λ, where $\{E^x_\lambda; \lambda \in \mathbb{R}\}$ is the spectral family of x ([6], [14]).

A complete (resp. conditionally complete) lattice L is a partially ordered set (poset) in which every subset (resp. bounded subset) S has a join $\bigvee^L S$, and a meet $\bigwedge^L S$. Let $L' \subset L$ be two complete lattices, and $S \subset L'$; by convention, we write $\bigvee S$ for $\bigvee^L S$. An isomorphism of posets is an order preserving injective map with order preserving inverse. Let L be a complete lattice, and $L' \subset L$. In general, the joins or meets in L' do not coincide with the ones in L, even when L' is a complete lattice; however, $\bigvee^{L'} S \geq \bigvee^L S$ for any $S \subset L'$ having a join in L'. A map $\gamma : L' \to L$ between complete lattices is sup-preserving if $\gamma(\bigvee S) = \bigvee\{\gamma(x); x \in S\}$ for all sets $S \subset L'$; the image of such a map is a complete lattice in which the joins coincide with the joins in L; the set $\mathrm{Sup}(L', L)$ of such maps is a complete lattice. Any surjective isomorphism of complete lattices is sup-preserving. We refer to [5] and [11] for more details.

2. Complete lattices of extended self-adjoint operators

We introduce here the notion of extended self-adjoint operator. By means of suitable isomorphisms, we identify complete lattices of such operators with complete lattices of sup-preserving maps (Theorem 2.3). The generalization of Olson's results is a direct consequence (Corollary 2.4).

Definition 2.1. Let $\mathcal{M}$ be a von Neumann algebra acting on some Hilbert space $\mathcal{H}$, and $\mathcal{P}$ the set of its projections. An *extended* self-adjoint operator x affiliated with $\mathcal{M}$ is a continuous from the right family $\{E^x_\lambda; \lambda \in \mathbb{R}\} \subset \mathcal{P}$. The *extended self-adjoint part* of $\mathcal{M}$ is the set $\mathcal{M}_{\overline{\mathbb{R}}}$ of all such operators provided with the *spectral order*: $x \preceq y$ if $E^x_\lambda \geq E^y_\lambda$ for all reals λ. The *spectrum* of x is the set $\sigma(x)$ of reals λ such that $E^x_{\lambda-\varepsilon} \neq E^x_{\lambda+\varepsilon}$ for all $\varepsilon > 0$. The *extended spectrum* $\sigma_{\overline{\mathbb{R}}}(x)$ is the set $\sigma(x)$ to which is added $-\infty$ (resp. $+\infty$) if $0 \neq \bigwedge_{\lambda\in\mathbb{R}} E^x_\lambda$ (resp. $1 \neq \bigvee_{\lambda\in\mathbb{R}} E^x_\lambda$). An element $x \in \mathcal{M}_{\overline{\mathbb{R}}}$ is *bounded* if $\sigma_{\overline{\mathbb{R}}}(x) \subset [-a, a]$ for some real a.

The above definition of the spectral order extends the one for bounded self-adjoint operators; this order coincides with the usual order on $\mathcal{P}$ and on commuting elements of $\mathcal{M}$ ([15]).

Let us introduce some notations. For any $x \in \mathcal{M}_{\overline{\mathbb{R}}}$ and all reals $a < b$, we put $E^x_{]-\infty,b[} = \bigvee_{\mu<b} E^x_\mu$, $E^x_{]a,b[} = E^x_{]-\infty,b[} \wedge (1 - E^x_a)$, $E^x_{]a,b]} = E^x_b \wedge (1 - E^x_a)$, $E^x_{]a,+\infty[} = 1 - E^x_a$, $E^x_{[a,+\infty[} = 1 - E^x_{]-\infty,a[}$. The elements $x + a1$ and $-x$ in $\mathcal{M}_{\overline{\mathbb{R}}}$ are respectively defined by $E^{x+a1}_\lambda = E^x_{\lambda-a}$ and $E^{-x}_\lambda = E^x_{[-\lambda,+\infty[}$ for all reals λ. For any sets $\mathcal{G} \subset \mathcal{P}$ and $L \subset \overline{\mathbb{R}}$, we denote by $\mathcal{M}^{\mathcal{G}}_L$ the set of elements $x \in \mathcal{M}_{\overline{\mathbb{R}}}$ such that $E^x_{]-\infty,\lambda[} \in \mathcal{G}$ for each real λ, and $\sigma_{\overline{\mathbb{R}}}(x) \subset L$. We shall omit the symbol $\mathcal{G}$ when $\mathcal{G} = \mathcal{P}$.

Then, $x \in \mathcal{M}_{]-\infty,+\infty]}$ if and only if there exists a self-adjoint operator T_x on some closed subspace $\mathcal{H}_x \subset \mathcal{H}$; note that $E^x_\lambda(\mathcal{H}) = E^{T_x}_\lambda(\mathcal{H}_x)$ for each real λ; in particular, $T_{-x} = -T_x$ and $\sigma(x) = \sigma(T_x)$. By Lemma 2.2, it is easy to see that $x \in \mathcal{M}_{[0,+\infty]}$ if and only if T_x is positive; in other words, $\mathcal{M}_{[0,+\infty]} = \{x \in \mathcal{M}_{\overline{\mathbb{R}}}; \forall\lambda < 0, E^x_\lambda = 0\}$ is the extended positive part of $\mathcal{M}$ in the sense of Haagerup ([12], Theorem 1.5); $x \in \mathcal{M}_{\mathbb{R}}$ if and only if $\mathcal{H}_x = \mathcal{H}$, that is $\mathcal{M}_{\mathbb{R}}$ is the set of self-adjoint operators on $\mathcal{H}$ affiliated with $\mathcal{M}$; x is bounded if and only if $\mathcal{H}_x = \mathcal{H}$ and T_x is bounded, that is the bounded part of $\mathcal{M}_{\overline{\mathbb{R}}}$ is the self-adjoint part of $\mathcal{M}$. In the rest of the paper, we identify x with T_x for any $x \in \mathcal{M}_{]-\infty,+\infty]}$.

Lemma 2.2. *Let $\psi : \mathcal{M}_{\overline{\mathbb{R}}} \to \overline{\mathbb{R}}^{\mathcal{P}}$ defined by $\psi(x)(p) = \inf\{\lambda \in \mathbb{R}; p \leq E^x_\lambda\}$. For each $x \in \mathcal{M}_{\overline{\mathbb{R}}}$, $p \in \mathcal{P}$, and $\lambda_0 \in \mathbb{R}$, the following properties hold:*

(a)

$$\sup\{\lambda \in \mathbb{R}; \forall \varepsilon > 0, pE^x_{]\lambda-\varepsilon,\lambda+\varepsilon]} \neq 0\} \leq \psi(x)(p). \tag{2.1}$$

If $\psi(x)(p)$ is finite, then the equality holds, the L.H.S. is a maximum, and the R.H.S. is a minimum; if moreover $p = E^x_{\lambda_0}$, then $p = E^x_{\psi(x)(p)}$ and $\psi(x)(p) = \min\{\mu \in \mathbb{R}; p = E^x_\mu\}$. If $\psi(x)(p) > -\infty$, then $\psi(x)(p) = \inf\{\lambda \in \sigma(x); p \leq E^x_\lambda\}$, which is a minimum if $\psi(x)(p)$ is finite.

(b) *$\lambda_0 \in \sigma(x)$ if and only if $\psi(x)(E^x_{\lambda_0}) = \lambda_0$. Moreover, $\sigma(x) = \emptyset$ if and only if $\psi(x)$ is $\{-\infty, +\infty\}$-valued.*

(c) *The following statements are equivalent:*
   (i) *$x \in \mathcal{M}$*
   (ii) *$\psi(x)_{|\mathcal{P}\setminus\{0\}}$ is real-valued and bounded.*
   (iii) *$\psi(x)_{|\mathcal{P}\setminus\{0\}}$ is $\sigma(x)$-valued and bounded.*

(d) *For any $\mathcal{G} \subset \mathcal{P}$ and each $x \in \mathcal{M}^{\mathcal{G}}_{\overline{\mathbb{R}}}$, $\psi(x)(p) = \inf_{q \geq p, q \in \mathcal{G}} \psi(x)(q)$.*

(e) *$\psi(x + \lambda_0 1) = \psi(x) + \lambda_0$, and $\psi(\lambda_0 x)_{|\mathcal{P}\setminus\{0\}} = \lambda_0 \psi(x)_{|\mathcal{P}\setminus\{0\}}$ if moreover $\lambda_0 \geq 0$ and $x \in \mathcal{M}$.*

Proof. (a) Let l and r denote respectively the L. H. S. and R. H. S. of (2.1). Since $\{E^x_\lambda; \lambda \in \mathbb{R}\}$ is continuous from the right, r is finite implies $p \leq E^x_r$ and r is a minimum. Suppose that $r < l$. There exists a real λ such that $r < \lambda < l$ and $pE^x_{]\lambda-\varepsilon,\lambda+\varepsilon]} \neq 0$ for all $\varepsilon > 0$, which implies $p \not\leq E^x_{\lambda-\varepsilon}$ for all $\varepsilon > 0$, and the contradiction if $r = -\infty$; if r is finite, then $p \leq E^x_r \leq E^x_{\lambda-\varepsilon}$ for ε sufficiently small gives the contradiction; thus, (2.1) holds. Suppose that r is finite. If $l < r$, then $pE^x_{]r-\varepsilon_0,r+\varepsilon_0]} = 0$ for some $\varepsilon_0 > 0$, and $p \leq E^x_{r-\varepsilon_0}$ (since $p \leq E^x_{r+\varepsilon_0}$) contradicts the definition of r; thus $l = r$; if l is not a maximum, then $pE^x_{]l-\varepsilon_0,l+\varepsilon_0]} = 0 = pE^x_{]r-\varepsilon_0,r+\varepsilon_0]}$ for some $\varepsilon_0 > 0$, and $p \leq E^x_{r-\varepsilon_0}$ (since $p \leq E^x_r$) contradicts the definition of r; if moreover $p = E^x_{\lambda_0}$, then

$$\psi(x)(p) = \min\{\mu \in \mathbb{R}; p \leq E^x_\mu\} \leq \inf\{\mu \in \mathbb{R}; p = E^x_\mu\} \leq \lambda_0,$$

which implies $p = E^x_{\psi(x)(p)}$, whence $\psi(x)(p) = \min\{\mu \in \mathbb{R}; p = E^x_\mu\}$. The last assertion is a direct consequence. The proofs of $(b), (c), (d)$ follow easily from (a), and are left to the reader. ☐

When $L \subset \overline{\mathbb{R}}$ has a bottom element 0_L, we define $\psi^{\mathcal{G}}_L : \mathcal{M}^{\mathcal{G}}_L \to L^{\mathcal{G}}$ by $\psi^{\mathcal{G}}_L(x)(p) = \inf\{\lambda \in \sigma(x); p \leq E^x_\lambda\}$ if $\psi(x)(p) > -\infty$ (where ψ is the map of Lemma 2.2), and $\psi^{\mathcal{G}}_L(x)(p) = 0_L$ otherwise. Since $\psi^{\mathcal{G}}_{L\cup\{-\infty\}}(x) = \psi(x)_{|\mathcal{G}}$ for all $x \in \mathcal{M}^{\mathcal{G}}_L$ by Lemma 2.2, we have in particular $\psi^{\mathcal{P}}_{\overline{\mathbb{R}}} = \psi$; we then shall omit the symbol $\mathcal{G}$ (resp. L) when $\mathcal{G} = \mathcal{P}$ (resp. $L = \overline{\mathbb{R}}$).

Theorem 2.3. *Let $\mathcal{G} \subset \mathcal{P}$ be a complete lattice containing 1 in which the joins coincide with the joins in $\mathcal{P}$, and let $L \subset \overline{\mathbb{R}}$ be a nonempty compact set. Then,*

(a) *$\mathcal{M}^{\mathcal{G}}_L$ is a complete lattice.*

(b) *$\psi^{\mathcal{G}}_L$ is an isomorphism of complete lattices from $\mathcal{M}^{\mathcal{G}}_L$ onto $\mathrm{Sup}(\mathcal{G}, L)$.*

(c) *For each $\gamma \in \mathrm{Sup}(\mathcal{G}, \overline{\mathbb{R}})$ and each real λ, we have $E^{(\psi^{\mathcal{G}})^{-1}(\gamma)}_\lambda = \bigwedge_{\mu>\lambda} E'_\mu$, where $E'_\mu = \bigvee\{p \in \mathcal{G}; \gamma(p) \leq \mu\}$ for each real μ.*

Proof. First step: the case $L = \overline{\mathbb{R}}$. Let $\gamma \in \mathrm{Sup}(\mathcal{G}, \overline{\mathbb{R}})$. For each real μ, we define the projection $E'_\mu = \bigvee\{p \in \mathcal{G}; \gamma(p) \leq \mu\}$. Then, $E'_{\mu_1} \leq E'_{\mu_2}$ when $\mu_1 \leq \mu_2$, and the family $\{E_\lambda = \bigwedge_{\mu>\lambda} E'_\mu; \lambda \in \mathbb{R}\}$ is a continuous from the right family of projections in $\mathcal{M}$. For each λ, μ, ε in $\mathbb{R}$ with $\lambda - \varepsilon < \mu < \lambda$, we have $E_{\lambda-\varepsilon} \leq E'_\mu \leq E_{]-\infty,\lambda[}$ so that $E_{]-\infty,\lambda[} = \bigvee_{\mu<\lambda} E'_\mu \in \mathcal{G}$. Put $\phi^{\mathcal{G}}(\gamma) = \{E_\lambda; \lambda \in \mathbb{R}\}$ and get a map $\phi^{\mathcal{G}} : \mathrm{Sup}(\mathcal{G}, \overline{\mathbb{R}}) \to \mathcal{M}^{\mathcal{G}}_{\overline{\mathbb{R}}}$. Note that for each $x \in \mathcal{M}^{\mathcal{G}}_{\overline{\mathbb{R}}}$, the following properties hold:

(i) $p \leq E^x_{\psi^{\mathcal{G}}(x)(p)}$ for all $p \in \mathcal{G}$ with $\psi^{\mathcal{G}}(x)(p) \in \mathbb{R}$ (by Lemma 2.2).

(ii) $\psi^{\mathcal{G}}(x)(E^x_{]-\infty,\lambda[}) \leq \lambda$ for each real λ ($\psi^{\mathcal{G}}(x)(E^x_{]-\infty,\lambda[})$ is well defined since $x \in \mathcal{M}^{\mathcal{G}}_{\overline{\mathbb{R}}}$).

Let $x = \{E^x_\lambda; \lambda \in \mathbb{R}\} \in \mathcal{M}^{\mathcal{G}}_{\overline{\mathbb{R}}}$ and $\{p_i; i \in I\} \subset \mathcal{G}$. Since $\psi^{\mathcal{G}}(x)$ is clearly increasing, we have $\psi^{\mathcal{G}}(x)(\bigvee_{i\in I} p_i) \geq \sup_{i\in I} \psi^{\mathcal{G}}(x)(p_i)$, and to prove the converse inequality, we can assume $\sup_{i\in I} \psi^{\mathcal{G}}(x)(p_i) \in \mathbb{R}$. By (i), $p_i \leq E^x_{\psi^{\mathcal{G}}(x)(p_i)}$ for all $i \in I$, and so $\bigvee_{i\in I} p_i \leq E^x_{\sup_{i\in I} \psi^{\mathcal{G}}(x)(p_i)}$. By (ii), $\psi^{\mathcal{G}}(x)(\bigvee_{i\in I} p_i) \leq \psi^{\mathcal{G}}(x)(E^x_{]-\infty,\sup_{i\in I} \psi^{\mathcal{G}}(x)(p_i)+\varepsilon[}) \leq \sup_{i\in I} \psi^{\mathcal{G}}(x)(p_i) + \varepsilon$ for all $\varepsilon > 0$, so that $\psi^{\mathcal{G}}(x)(\bigvee_{i\in I} p_i) \leq \sup_{i\in I} \psi^{\mathcal{G}}(x)(p_i)$ and $\psi^{\mathcal{G}}(x) \in \mathrm{Sup}(\mathcal{G}, \overline{\mathbb{R}})$.

We will prove that $\gamma = \psi^{\mathcal{G}} \circ \phi^{\mathcal{G}}(\gamma)$ for all $\gamma \in \mathrm{Sup}(\mathcal{G}, \overline{\mathbb{R}})$. Let $\gamma \in \mathrm{Sup}(\mathcal{G}, \overline{\mathbb{R}})$, $p \in \mathcal{G}$ and $\phi^{\mathcal{G}}(\gamma) = \{E_\lambda; \lambda \in \mathbb{R}\}$.

We first show that $\gamma(p) \geq \psi^{\mathcal{G}} \circ \phi^{\mathcal{G}}(\gamma)(p)$. We can suppose $\gamma(p) < +\infty$, which implies $\inf\{\lambda \in \mathbb{R}; p \leq E_\lambda\} < +\infty$. Let $\lambda \in \mathbb{R}$ such that $p(E_{\lambda+\varepsilon} - E_{\lambda-\varepsilon}) \neq 0$ for all $\varepsilon > 0$. Then $p \not\leq E_{\lambda-\varepsilon}$, and therefore $p \not\leq E'_{\lambda-\varepsilon}$ which implies $\gamma(p) \geq \lambda - \varepsilon$ for all $\varepsilon > 0$. Thus, $\gamma(p) \geq \lambda$, and $\gamma(p) \geq \psi^{\mathcal{G}} \circ \phi^{\mathcal{G}}(\gamma)(p)$ by Lemma 2.2.

We show now that $\gamma(p) \leq \psi^{\mathcal{G}} \circ \phi^{\mathcal{G}}(\gamma)(p)$. Suppose that $\gamma(p) = +\infty$ and $\psi^{\mathcal{G}} \circ \phi^{\mathcal{G}}(\gamma)(p) < +\infty$. Then, $p \leq E_\lambda \leq \bigwedge_{\mu>\lambda} E'_\mu$ for some real λ with $\gamma(E'_\mu) \leq \mu$, which gives the contradiction. Thus, $\gamma(p) = +\infty$ implies $\psi^{\mathcal{G}} \circ \phi^{\mathcal{G}}(\gamma)(p) = +\infty$. Suppose that $\psi^{\mathcal{G}} \circ \phi^{\mathcal{G}}(\gamma)(p) < \gamma(p) < +\infty$, and note that $\gamma(p) \in \sigma(\phi^{\mathcal{G}}(\gamma))$ (otherwise, there exists $\varepsilon > 0$ such that $E_{\gamma(p)-\varepsilon} = E_{\gamma(p)+\varepsilon}$; let μ_1, μ_2 such that $\gamma(p) - \varepsilon < \mu_1 < \gamma(p) < \mu_2 < \gamma(p) + \varepsilon$, and obtain $E_{\gamma(p)-\varepsilon} \leq E'_{\mu_1} \leq E'_{\mu_2} \leq E_{\gamma(p)+\varepsilon}$ which implies $E'_{\mu_1} = E'_{\mu_2}$; but $p \leq E'_{\mu_2}$, and $\gamma(E'_{\mu_1}) \leq \mu_1$ implies $p \not\leq E'_{\mu_1}$, which gives the contradiction). By Lemma 2.2, we have $p(E_{\gamma(p)+\varepsilon'} - E_{\gamma(p)-\varepsilon'}) = 0$ for some $\varepsilon' > 0$, with $E_{\gamma(p)+\varepsilon'} - E_{\gamma(p)-\varepsilon'} \neq 0$. Since $p \leq E'_{\gamma(p)+\varepsilon'} \leq E_{\gamma(p)+\varepsilon'}$, we get $p \leq E_{\gamma(p)-\varepsilon'}$. Choose μ with $\gamma(p) - \varepsilon' < \mu < \gamma(p)$ and obtain $\gamma(p) \leq \gamma(E'_\mu) \leq \mu$, which gives the contradiction. Thus, $\gamma(p) \leq \psi^{\mathcal{G}} \circ \phi^{\mathcal{G}}(\gamma)(p)$.

We prove now that $\psi^{\mathcal{G}}$ is injective. Let x_1 and x_2 in $\mathcal{M}^{\mathcal{G}}_{\overline{\mathbb{R}}}$ such that $\gamma = \psi^{\mathcal{G}}(x_1) = \psi^{\mathcal{G}}(x_2)$. If $\gamma(E^{x_1}_{]-\infty,\lambda+\varepsilon[}) > -\infty$, then $\gamma(E^{x_1}_{]-\infty,\lambda+\varepsilon[}) \in \mathbb{R}$ and for each real λ and $\varepsilon > 0$, we have

$$E^{x_1}_{]-\infty,\lambda+\varepsilon[} \leq E^{x_2}_{\gamma(E^{x_1}_{]-\infty,\lambda+\varepsilon[})} \leq E^{x_2}_{\lambda+\varepsilon}$$

by (i) and (ii), which gives $E^{x_1}_\lambda \leq E^{x_2}_\lambda$ by right continuity; since $E^{x_1}_\lambda \leq E^{x_2}_\lambda$ if $\gamma(E^{x_1}_{]-\infty,\lambda+\varepsilon[}) = -\infty$, in any case we have $E^{x_1}_\lambda \leq E^{x_2}_\lambda$, and by symmetry $E^{x_1}_\lambda = E^{x_2}_\lambda$.

We have proved that $\psi^{\mathcal{G}}$ is a bijection from $\mathcal{M}^{\mathcal{G}}_{\overline{\mathbb{R}}}$ to $\mathrm{Sup}(\mathcal{G}, \overline{\mathbb{R}})$, and (c) holds; (a) and (b) will follow once proved that $\psi^{\mathcal{G}}$ is a poset isomorphism. Let x_1 and x_2 in $\mathcal{M}^{\mathcal{G}}_{\overline{\mathbb{R}}}$. Suppose that $\psi^{\mathcal{G}}(x_1) \leq \psi^{\mathcal{G}}(x_2)$. For all reals λ and $\varepsilon > 0$, we have

$$\psi^{\mathcal{G}}(x_1)(E^{x_2}_{]-\infty,\lambda+\varepsilon[}) \leq \psi^{\mathcal{G}}(x_2)(E^{x_2}_{]-\infty,\lambda+\varepsilon[}) \leq \lambda + \varepsilon$$

by Lemma 2.2 applied to $\psi(x_2)$. If $\psi^{\mathcal{G}}(x_1)(E^{x_2}_{]-\infty,\lambda+\varepsilon[}) = -\infty$, then $E^{x_2}_{]-\infty,\lambda+\varepsilon[} \leq E^{x_1}_{\lambda}$ and $E^{x_2}_{\lambda} \leq E^{x_1}_{\lambda}$. If $\psi^{\mathcal{G}}(x_1)(E^{x_2}_{]-\infty,\lambda+\varepsilon[}) > -\infty$, then by Lemma 2.2 applied to $\psi^{\mathcal{G}}(x_1)$, there is some real λ' such that $E^{x_2}_{]-\infty,\lambda+\varepsilon[} \leq E^{x_1}_{\lambda'}$ and $\psi^{\mathcal{G}}(x_1)(E^{x_2}_{]-\infty,\lambda+\varepsilon[}) = \lambda'$, which implies

$$E^{x_2}_{]-\infty,\lambda+\varepsilon[} \leq E^{x_1}_{\lambda'} \leq E^{x_1}_{]-\infty,\lambda+\varepsilon[},$$

and $E^{x_2}_{\lambda} \leq E^{x_1}_{\lambda}$ by right continuity, i.e. $x_1 \preceq x_2$. Suppose that $x_1 \preceq x_2$ and $\psi^{\mathcal{G}}(x_2)(p) < \psi^{\mathcal{G}}(x_1)(p)$ for some $p \in \mathcal{G}$. By Lemma 2.2, there exists a real λ such that $p \leq E^{x_2}_{\lambda}$ and

$$\psi^{\mathcal{G}}(x_2)(p) \leq \lambda < \psi^{\mathcal{G}}(x_1)(p). \tag{2.2}$$

Since $x_1 \preceq x_2$, we have $p \leq E^{x_2}_{\lambda} \leq E^{x_1}_{\lambda}$ and $\psi^{\mathcal{G}}(x_1)(p) \leq \lambda$, which contradicts (2.2); thus $x_1 \preceq x_2$ implies $\psi^{\mathcal{G}}(x_1) \leq \psi^{\mathcal{G}}(x_2)$. The theorem is proved when $L = \overline{\mathbb{R}}$.

Second step: the general case. Put $L_- = L \cup \{-\infty\}$. Since the nonempty joins in L coincide with the nonempty joins in $\overline{\mathbb{R}}$, $\mathrm{Sup}(\mathcal{G}, L_-) \subset \mathrm{Sup}(\mathcal{G}, \overline{\mathbb{R}})$, and since any set in $\mathrm{Sup}(\mathcal{G}, L_-)$ has a join (which is the join in $\mathrm{Sup}(\mathcal{G}, \overline{\mathbb{R}})$), $\mathrm{Sup}(\mathcal{G}, L_-)$ is a complete lattice; consequently, $(\psi^{\mathcal{G}})^{-1}(\mathrm{Sup}(\mathcal{G}, L_-))$ is a complete lattice. For each $\gamma \in \mathrm{Sup}(\mathcal{G}, L_-)$, $\gamma = \psi^{\mathcal{G}}(x)$ for some $x \in \mathcal{M}^{\mathcal{G}}_{\overline{\mathbb{R}}}$ by the preceding case. Let $\lambda \in \sigma(x)$. By Lemma 2.2 (b) and (d), we have

$$\lambda = \inf_{q \geq E^x_{\lambda}, q \in \mathcal{G}} \psi(x)(q) = \inf_{q \geq E^x_{\lambda}, q \in \mathcal{G}} \psi^{\mathcal{G}}(x)(q),$$

so that $\lambda \in L$ since $\psi^{\mathcal{G}}(x)(q) \in L$. If $+\infty \in \sigma_{\overline{\mathbb{R}}}(x)$, then $\psi^{\mathcal{G}}(x)(1) = \psi(x)(1) = +\infty \in L$. It follows that $\sigma_{\overline{\mathbb{R}}}(x) \subset L_-$, hence $(\psi^{\mathcal{G}})^{-1}(\mathrm{Sup}(\mathcal{G}, L_-)) \subset \mathcal{M}^{\mathcal{G}}_{L_-}$. Since $\psi^{\mathcal{G}}(x)_{|\mathcal{G}\setminus\{0\}} = \psi(x)_{|\mathcal{G}\setminus\{0\}}$ is $\sigma_{\overline{\mathbb{R}}}(x)$-valued for each $x \in \mathcal{M}^{\mathcal{G}}$ by Lemma 2.2, we have $\psi^{\mathcal{G}}(\mathcal{M}^{\mathcal{G}}_{L_-}) \subset \mathrm{Sup}(\mathcal{G}, L_-)$, so that $(\psi^{\mathcal{G}})^{-1}(\mathrm{Sup}(\mathcal{G}, L_-)) = \mathcal{M}^{\mathcal{G}}_{L_-}$. Therefore, ($a$) and ($b$) hold for L_- since $\psi^{\mathcal{G}}_{L_-} = \psi^{\mathcal{G}}_{|\mathcal{M}^{\mathcal{G}}_{L_-}}$. It remains to prove the case where $-\infty \notin L$. Put $\mathrm{Sup}(\mathcal{G}, L_-)' = \{\gamma \in \mathrm{Sup}(\mathcal{G}, L_-); \forall p \neq 0, \gamma(p) \in L\}$. It is easy to verify that the map $\Delta : \mathrm{Sup}(\mathcal{G}, L_-)' \to \mathrm{Sup}(\mathcal{G}, L)$ defined by $\Delta(\gamma)(p) = \gamma(p)$ for $p \neq 0$ and $\Delta(\gamma)(0) = 0_L$ is a surjective poset isomorphism with converse map $\Delta^{-1}(\gamma)(p) = \gamma(p)$ for $p \neq 0$ and $\Delta^{-1}(\gamma)(0) = -\infty$; in particular, $\mathrm{Sup}(\mathcal{G}, L_-)'$ is a complete lattice. Note that $\psi^{\mathcal{G}}_L = \Delta \circ \psi^{\mathcal{G}}_{|\mathcal{M}^{\mathcal{G}}_L}$. For each $x \in \mathcal{M}^{\mathcal{G}}_L$, $\psi^{\mathcal{G}}(x)_{|\mathcal{G}\setminus\{0\}}$ is L-valued, and so $\psi^{\mathcal{G}}(x) \in \mathrm{Sup}(\mathcal{G}, L_-)'$ since $\psi^{\mathcal{G}}(x) \in \mathrm{Sup}(\mathcal{G}, L_-)$. Conversely, each $\gamma \in \mathrm{Sup}(\mathcal{G}, L_-)'$ has the form $\gamma = \psi^{\mathcal{G}}(x)$ for some $x \in \mathcal{M}^{\mathcal{G}}_{L_-}$; but $x \in \mathcal{M}^{\mathcal{G}}_L$ since $\psi^{\mathcal{G}}(x)_{|\mathcal{G}\setminus\{0\}}$ is L-valued. Therefore, $\psi^{\mathcal{G}}_{|\mathcal{M}^{\mathcal{G}}_L}$ is a poset isomorphism onto $\mathrm{Sup}(\mathcal{G}, L_-)'$; $\mathcal{M}^{\mathcal{G}}_L$ is then a complete lattice, and $\psi^{\mathcal{G}}_L$ an isomorphism of complete lattices as a composition of such maps. This proves (a) and (b) for L. □

In [15], M. P. Olson has shown that the self-adjoint part of $\mathcal{M}$ (i.e., $\mathcal{M}_{\mathbb{R}} \cap \mathcal{M}$) is a conditionally complete lattice, in which the joins and meets of a bounded set $\{x_i; i \in I\}$ are given respectively by the first and second equality of (b). These results are recovered in the following corollary by considering the bounded part of $\mathcal{M}^{\mathcal{G}}_{]a,b[}$ with $a = -\infty$, $b = +\infty$, and $\mathcal{G} = \mathcal{P}$.

Corollary 2.4. *Under the same hypotheses as Theorem 2.3, we have:*

(a) *Any isomorphism from L onto $\overline{\mathbb{R}}$ induces an isomorphism from $\mathcal{M}^{\mathcal{G}}_L$ onto $\mathcal{M}^{\mathcal{G}}_{\overline{\mathbb{R}}}$. If $L = [a, b]$ with a, b in $\overline{\mathbb{R}}$, then $\mathcal{M}^{\mathcal{G}}_{]a,b]}$, $\mathcal{M}^{\mathcal{G}}_{]a,b[}$, $\mathcal{M}^{\mathcal{G}}_{[a,b[}$, and their* *respective bounded parts are conditionally complete lattices, in which the nonempty joins and nonempty meets coincide with the ones in* $\mathcal{M}^{\mathcal{G}}_{\overline{\mathbb{R}}}$. *In particular,* $\mathcal{M}^{\mathcal{G}}_{\overline{\mathbb{R}}}$ *is a completion for* $\mathcal{M}^{\mathcal{G}}_{]a,b[}$, $\mathcal{M}^{\mathcal{G}}_{[-\infty,b[}$, $\mathcal{M}^{\mathcal{G}}_{]a,+\infty]}$, *and for their respective bounded parts.*

(b) *For any set* $\{x_i; i \in I\} \subset \mathcal{M}_{\overline{\mathbb{R}}}$ *and each real* λ,

$$E_\lambda^{\bigvee_{i\in I} x_i} = \bigwedge_{i\in I} E_\lambda^{x_i}, \quad E_\lambda^{\bigwedge_{i\in I} x_i} = \bigwedge_{\mu>\lambda}\bigvee_{i\in I} E_\mu^{x_i}, \quad E_{]-\infty,\lambda[}^{\bigwedge_{i\in I} x_i} = \bigvee_{i\in I} E_{]-\infty,\lambda[}^{x_i}.$$

(c) *For each* $x \in \mathcal{M}_{\overline{\mathbb{R}}}$,

$$x = \bigwedge_{a\in\mathbb{R}} x \vee a1 = \bigvee_{b\in\mathbb{R}} x \wedge b1 = \bigvee_{b\in\mathbb{R}}\bigwedge_{a\in\mathbb{R}}((x\wedge b1)\vee a1) = \bigwedge_{a\in\mathbb{R}}\bigvee_{b\in\mathbb{R}}((x\wedge b1)\vee a1).$$

(d) $\mathcal{M}^{\mathcal{G}}_{\{0,1\}} = \{1-p; p \in \mathcal{G}\}$.

Proof. (*a*) Any isomorphism h from L onto $\overline{\mathbb{R}}$ induces an isomorphism from $\mathrm{Sup}(\mathcal{G}, L)$ onto $\mathrm{Sup}(\mathcal{G}, \overline{\mathbb{R}})$ defined by $h \circ \gamma(p)$ for all $\gamma \in \mathrm{Sup}(\mathcal{G}, L)$ and $p \in \mathcal{G}$; the first assertion follows then from Theorem 2.3. Now, suppose $L = [-\infty, +\infty]$. For each bounded subset $\mathcal{S} \subset \mathcal{M}^{\mathcal{G}}_{\mathbb{R}}$, $\bigwedge\mathcal{S}$ and $\bigvee\mathcal{S}$ exist in $\mathcal{M}^{\mathcal{G}}_{\overline{\mathbb{R}}}$ with $x \preceq \bigwedge\mathcal{S} \preceq \bigvee\mathcal{S} \preceq y$ for some x and y in $\mathcal{M}^{\mathcal{G}}_{\mathbb{R}}$, that is $\psi(x) \le \psi(\bigwedge\mathcal{S}) \le \psi(\bigvee\mathcal{S}) \le \psi(y)$ by Theorem 2.3. Then, $\bigwedge_{\lambda\in\mathbb{R}} E_\lambda^{\bigwedge\mathcal{S}} = \bigwedge_{\lambda\in\mathbb{R}} E_\lambda^{\bigvee\mathcal{S}} = 0$ and $\bigvee_{\lambda\in\mathbb{R}} E_\lambda^{\bigwedge\mathcal{S}} = \bigvee_{\lambda\in\mathbb{R}} E_\lambda^{\bigvee\mathcal{S}} = 1$, and the second assertion holds for $\mathcal{M}^{\mathcal{G}}_{\mathbb{R}}$. Replace $\mathcal{M}^{\mathcal{G}}_{\mathbb{R}}$ by $\mathcal{M}^{\mathcal{G}}_{\mathbb{R}} \cap \mathcal{M}$ in the above proof, and get $\bigwedge\mathcal{S} \in \mathcal{M}^{\mathcal{G}}_{\mathbb{R}} \cap \mathcal{M}$ and $\bigvee\mathcal{S} \in \mathcal{M}^{\mathcal{G}}_{\mathbb{R}} \cap \mathcal{M}$ since for any $z \in \mathcal{M}^{\mathcal{G}}_{\overline{\mathbb{R}}}$, $z \in \mathcal{M}^{\mathcal{G}}_{\mathbb{R}} \cap \mathcal{M}$ if and only if $\psi(z)_{|\mathcal{P}\setminus\{0\}}$ is real-valued and bounded by Lemma 2.2; this proves the second assertion for $\mathcal{M}^{\mathcal{G}}_{\mathbb{R}} \cap \mathcal{M}$. The others cases are proved similarly by noting that $\lambda \in \sigma_{\overline{\mathbb{R}}}(z)\setminus\{+\infty\}$ if and only if $\psi(z)(p) = \lambda$ for some $p \in \mathcal{P}\setminus\{0\}$, and $+\infty \in \sigma_{\overline{\mathbb{R}}}(z)$ implies $\psi(z)(1) = +\infty$. The last assertion follows by noting that when it exists, the top (resp. bottom) element of any of the mentioned lattices coincides with the one of $\mathcal{M}^{\mathcal{G}}_{\overline{\mathbb{R}}}$.

(*b*) Let $\lambda \in \mathbb{R}$. Since $\psi(x_i)(\bigwedge_{i\in I} E_\lambda^{x_i}) \le \lambda$ for all $i \in I$, we have $\psi(\bigvee_{i\in I} x_i)(\bigwedge_{i\in I} E_\lambda^{x_i}) \le \lambda$, and so $\bigwedge_{i\in I} E_\lambda^{x_i} \le (E^{\bigvee_{i\in I} x_i})'_\mu$ for all $\mu > \lambda$ by Theorem 2.3, which implies $\bigwedge_{i\in I} E_\lambda^{x_i} \le E_\lambda^{\bigvee_{i\in I} x_i}$, and the first equality holds (the converse inequality is obvious). Define $z \in \mathcal{M}_{\overline{\mathbb{R}}}$ by $E_\lambda^z = \bigwedge_{\mu>\lambda}\bigvee_{i\in I} E_\mu^{x_i}$, and note that $E_\lambda^z \ge \bigvee_{i\in I} E_\lambda^{x_i}$, and $E_\lambda^z \le \bigvee_{i\in I} E_\mu^{x_i}$ for all reals $\mu > \lambda$. Then, $E_\lambda^{\bigwedge_{i\in I} x_i} \ge \bigvee_{i\in I} E_\lambda^{x_i}$ is equivalent to $\bigwedge_{\mu>\lambda} E_\mu^{\bigwedge_{i\in I} x_i} \ge \bigvee_{i\in I} E_\lambda^{x_i}$, and $E_\mu^{\bigwedge_{i\in I} x_i} \ge \bigvee_{i\in I} E_\mu^{x_i}$ implies $E_\lambda^{\bigwedge_{i\in I} x_i} \ge E_\lambda^z$, that is $\bigwedge_{i\in I} x_i = z$, and the second equality holds. For each $\mu < \lambda$ and each $i \in I$, we have by the second equality,

$$E_\mu^{x_i} \le E_\mu^{\bigwedge_{i\in I} x_i} = \bigwedge_{\nu>\mu}\bigvee_{i\in I} E_\nu^{x_i} \le \bigwedge_{\mu<\nu<\lambda}\bigvee_{i\in I} E_\nu^{x_i} \le \bigvee_{i\in I} E_{]-\infty,\lambda[}^{x_i},$$

so that

$$\bigvee_{i\in I}\bigvee_{\mu<\lambda} E_\mu^{x_i} = \bigvee_{i\in I} E_{]-\infty,\lambda[}^{x_i} = \bigvee_{\mu<\lambda} E_\mu^{\bigwedge_{i\in I} x_i} = E_{]-\infty,\lambda[}^{\bigwedge_{i\in I} x_i},$$

and the third equality holds.

(*c*) By (*b*), we have $E_\mu^{x\vee a1} = E_\mu^x$ if $\mu \ge a$, and $E_\mu^{x\vee a1} = 0$ if $\mu < a$, which implies

$$E_\mu^{\bigwedge_{a\in\mathbb{R}} x\vee a1} = \bigwedge_{\nu>\mu}\bigvee_{a\in\mathbb{R}} E_\nu^{x\vee a1} = E_\mu^x, \tag{2.3}$$

and the first equality holds. Since $E_\mu^{x\wedge b1} = E_\mu^x$ if $\mu \le b$, and $E_\mu^{x\wedge b1} = 1$ if $\mu > b$, we get

$$E_\mu^{\bigvee_{b\in\mathbb{R}} x\wedge b1} = \bigwedge_{b\in\mathbb{R}} E_\mu^{x\wedge b1} = E_\mu^x, \tag{2.4}$$

and the second equality holds. Put $y = x \wedge b1$, and get by (2.3) and (2.4),

$$E_\mu^{\bigvee_{b\in\mathbb{R}}(\bigwedge_{a\in\mathbb{R}}(y\vee a1))} = \bigwedge_{b\in\mathbb{R}} E_\mu^{\bigwedge_{a\in\mathbb{R}}(y\vee a1)} = \bigwedge_{b\in\mathbb{R}} E_\mu^y = E_\mu^x,$$

so that the third equality holds. The last one is proved similarly since $(x\wedge b1)\vee a1 = (x \vee a1) \wedge b1$.

(*d*) Clearly $\psi_{|\{1-p;p\in\mathcal{G}\}}$ is injective with values in $\mathrm{Sup}(\mathcal{G}, \{0,1\})$. Let $\gamma \in \mathrm{Sup}(\mathcal{G}, \{0,1\})$. Define $p_0 = \bigvee\{p \in \mathcal{G}; \gamma(p) = 0\}$ and note that $p_0 \in \mathcal{G}$ with $\gamma(p_0) = 0$. For each $p \in \mathcal{G}$, $\psi(1 - p_0)(p) = 0$ if and only if $p \le p_0$ if and only if $\gamma(p) = 0$, hence $\psi(1 - p_0) = \gamma$ and $\psi_{|\{1-p;p\in\mathcal{G}\}}$ is surjective. □

3. Extended q-semicontinuity and q-upper regularization

In this section, we take $\mathcal{M} = A''$ the universal enveloping von Neumann algebra of a C*-algebra A, and $\mathcal{G}$ the set of open projections. We define the q-upper regularization map, and study its properties via the isomorphisms ψ and $\psi^{\mathcal{G}}$ (Theorems 3.3 and 3.4). These ones look like noncommutative versions of well-known properties for functions; we will show in the next section, that there are true generalizations, and not only formal similarities.

Note that $x \in A''^{\mathcal{G}}_{[-\infty,+\infty[}$ (resp. $x \in A''^{\mathcal{G}}_{\overline{\mathbb{R}}} \cap A''$) if and only if x is a unbounded (resp. bounded) q-upper semicontinuous operator in the sense of [14]. Since $A''^{\mathcal{G}}_{\overline{\mathbb{R}}}$ is a complete lattice by Theorem 2.3, we can give the following definition.

Definition 3.1. Let $x \in A''_{\overline{\mathbb{R}}}$. Then, x is *extended q-upper semicontinuous* (resp. *extended q-lower semicontinuous*, *extended q-continuous*) if $x \in A''^{\mathcal{G}}_{\overline{\mathbb{R}}}$ (resp. $-x \in A''^{\mathcal{G}}_{\overline{\mathbb{R}}}$, $x \in A''^{\mathcal{G}}_{\overline{\mathbb{R}}}$ and $-x \in A''^{\mathcal{G}}_{\overline{\mathbb{R}}}$). The element $\overline{x} = \bigwedge^{A''^{\mathcal{G}}_{\overline{\mathbb{R}}}}\{y \in A''^{\mathcal{G}}_{\overline{\mathbb{R}}}; y \succeq x\}$ is the *q-upper regularization* of x.

The closure of a projection $p \in \mathcal{P}$ has been introduced in [1] as the least closed projection greater than p; it coincides with $\overline{p}$ as shows the next proposition (note that $A''^{\mathcal{G}}_{\{0,1\}}$ is the complete lattice of closed projections by Corollary 2.4). Moreover, the meet in $A''_{\overline{\mathbb{R}}}$ of any family of extended q-upper semicontinuous operators is extended q-upper semicontinuous, so that $x \preceq \overline{x}$, and the map $x \mapsto \overline{x}$ is then a closure operator on $A''_{\overline{\mathbb{R}}}$. The next proposition following easily from Corollary 2.4, the proof is left to the reader.

Proposition 3.2. *For any $\mathcal{S} \subset A''^{\mathcal{G}}_{\overline{\mathbb{R}}}$, if the meet of $\mathcal{S}$ exists in one of the following lattices: $A''^{\mathcal{G}}_{\overline{\mathbb{R}}}$, $A''^{\mathcal{G}}_{\mathbb{R}} \cap A''$, $A''^{\mathcal{G}}_{\{0,1\}}$, $A''_{\mathbb{R}}$, $A''_{\mathbb{R}} \cap A''$, $A''_{\{0,1\}}$, then it exists in any other containing $\mathcal{S}$, and coincides with the meet of $\mathcal{S}$ in $A''^{\mathcal{G}}_{\overline{\mathbb{R}}}$.*

The following theorem gives various characterizations of the q-upper regularization map, in terms of ψ and $\psi^{\mathcal{G}}$. We will prove in Corollary 4.4 that they are generalizations (even in the commutative case) of well-known characterizations of the upper regularization of $\overline{\mathbb{R}}$-valued functions on a locally compact Hausdorff space.

Theorem 3.3. *For each $x \in A''_{\overline{\mathbb{R}}}$ and each $p \in \mathcal{P}$, the following properties hold:*

(i) *$\overline{x}$ is the unique extended q-upper semicontinuous operator y satisfying*

$$\psi^{\mathcal{G}}(y) = \psi(x)_{|\mathcal{G}}; \tag{3.1}$$

(ii) *$\overline{x} = \bigvee\{y \in A''_{\overline{\mathbb{R}}}; \psi(y)_{|\mathcal{G}} = \psi(x)_{|\mathcal{G}}\}$;*

(iii) *$\psi(\overline{x})(p) = \inf_{q \geq p, q \in \mathcal{G}} \psi(x)(q)$.*

Proof. Let $x \in A''_{\overline{\mathbb{R}}}$. Since the join in $\mathcal{P}$ of any family of open projections is an open projection, $\psi(x)_{|\mathcal{G}} \in \mathrm{Sup}(\mathcal{G}, \overline{\mathbb{R}})$, and $(\psi^{\mathcal{G}})^{-1}(\psi(x)_{|\mathcal{G}}) \in A''^{\mathcal{G}}_{\overline{\mathbb{R}}}$ by Theorem 2.3. Put $x_0 = (\psi^{\mathcal{G}})^{-1}(\psi(x)_{|\mathcal{G}})$. By Lemma 2.2 (d), we have $\psi(x_0)(p) = \inf_{q \geq p, q \in \mathcal{G}} \psi(x_0)(q)$ for all $p \in \mathcal{P}$, and since $\psi^{\mathcal{G}}(x_0)(q) = \psi(x)(q)$ for each $q \in \mathcal{G}$, it follows that $\psi(x_0)(p) \geq \psi(x)(p)$, i.e. $x_0 \succeq x$ by Theorem 2.3. For all $z \in A''^{\mathcal{G}}_{\overline{\mathbb{R}}}$ with $z \succeq x$, we have $\psi^{\mathcal{G}}(z) = \psi(z)_{|\mathcal{G}} \geq \psi(x)_{|\mathcal{G}}$, and so $(\psi^{\mathcal{G}})^{-1} \circ \psi^{\mathcal{G}}(z) = z \succeq x_0$. We then have $x_0 = \overline{x}$ so that

$$\psi^{\mathcal{G}}(\overline{x}) = \psi(x)_{|\mathcal{G}}, \tag{3.2}$$

and $\overline{x}$ satisfies (3.1). If $y \in A''^{\mathcal{G}}_{\overline{\mathbb{R}}}$ satisfies (3.1), then $y = \overline{y} = (\psi^{\mathcal{G}})^{-1}(\psi(y)_{|\mathcal{G}}) = (\psi^{\mathcal{G}})^{-1}(\psi(x)_{|\mathcal{G}}) = \overline{x}$ by (3.2), and (i) holds. Put $z = \bigvee\{y \in A''_{\overline{\mathbb{R}}}; \psi(y)_{|\mathcal{G}} = \psi(x)_{|\mathcal{G}}\}$. By Theorem 2.3, $\psi(z)(q) = \sup\{\psi(y)(q); y \in A''_{\overline{\mathbb{R}}}, \psi(y)_{|\mathcal{G}} = \psi(x)_{|\mathcal{G}}\}$, so that $\psi(z)_{|\mathcal{G}} = \psi(x)_{|\mathcal{G}}$, which gives $\overline{z} = \overline{x}$ by (i). Apply (3.2) to z, and get $\overline{z} = z$ by definition of z since $\overline{z} \succeq z$, which gives (ii). For each $p \in \mathcal{P}$, $\psi(\overline{x})(p) = \inf_{q \geq p, q \in \mathcal{G}} \psi(\overline{x})(q)$ by Lemma 2.2 (d), and since $\psi(\overline{x})_{|\mathcal{G}} = \psi(x)_{|\mathcal{G}}$ by (i), (iii) holds. □

The following theorem is a noncommutative version of the Dini-Cartan lemma ([9]), as will establish Corollary 4.5. Suppose A unital, and let $(x_i)_{i \in I}$ be a decreasing net (with respect to the spectral order) of bounded q-upper semicontinuous operators satisfying $\bigwedge_{i \in I} x_i = 0$. Since $\psi(x)(1) = ||x||$ for all $x \in A''_{\mathbb{R}} \cap A$ by Lemma 2.2 (a), (3.3) with $p = 1$ gives $\inf_{i \in I} ||x_i|| = 0$. If moreover each $x_i = p_i$ is a projection (and thus a compact one), we get $p_{i_0} = 0$ for some $i_0 \in I$, and we recover a well known result of Akemann ([1], Proposition II. 10).

Theorem 3.4. *Let $(x_i)_{i \in I}$ be a decreasing net of extended q-upper semicontinuous operators. Then, for all compact projections p commuting with all the $E^{x_i}_\lambda$ ($i \in I, \lambda \in \mathbb{R}$), we have*

$$\psi(\bigwedge_{i \in I} x_i)(p) = \inf_{i \in I} \psi(x_i)(p). \tag{3.3}$$

Proof. By means of an increasing homeomorphism from $[0, 1]$ onto $\overline{\mathbb{R}}$ (which is a surjective isomorphism of complete lattices), we can suppose $\{x_i; i \in I\} \subset A''^{\mathcal{G}}_{[0,1]}$ by Corollary 2.4. Put $x = \bigwedge_{i \in I} x_i$, and let $p \in \mathcal{P}$. Clearly, $\psi(x)(p) \leq \inf_{i \in I} \psi(x_i)(p)$. Suppose $\psi(x)(p) < s < \inf_{i \in I} \psi(x_i)(p)$ for some real s. Then,

$$\inf\{\lambda \in \mathbb{R}; p \leq E^x_\lambda\} < s < \inf_{i \in I} \inf\{\lambda \in \mathbb{R}; p \leq E^{x_i}_\lambda\} \tag{3.4}$$

and so there exists a real $\lambda < s$ such that $p \leq E^x_\lambda$. Since $E^x_{]\lambda,+\infty[} = \bigvee_{\mu > \lambda} \bigwedge_{i \in I} E^{x_i}_{]\mu,+\infty[}$ by Corollary 2.4, we have $p \wedge \bigwedge_{i \in I} E^{x_i}_{]\mu,+\infty[} = 0$ for all $\mu > \lambda$, hence $p \wedge \bigwedge_{i \in I} E^{x_i}_{[s,+\infty[} = 0$ by taking $\mu < s$. If p is compact, then $p \wedge E^{x_{i_0}}_{[s,+\infty[} = 0$ for some x_{i_0} ([2], Proposition 1.1.2), that is $pE^{x_{i_0}}_{[s,+\infty[} = 0$ if moreover p commutes with all the $E^{x_i}_\lambda$. But (3.4) implies $p \not\leq E^{x_{i_0}}_s$ and the contradiction. □

4. An operator-theoretic representation for $\overline{\mathbb{R}}^X$

The aim of this section is to prove that the properties of the q-upper regularization map given in the preceding section generalize the classical ones; that is, properties for functions on X must be recovered by taking A commutative with spectrum X in Theorems 3.3 and 3.4; this will be obtained in Corollaries 4.4 and 4.5. For that purpose, we need to represent any $\overline{\mathbb{R}}$-valued function on X as a true operator in such a way that this representation (say Π) coincides with the known notions of q-upper semicontinuous operators (bounded or unbounded), and satisfies $\overline{\Pi(f)} = \Pi(\overline{f})$; this is achieved in Theorem 4.2, where moreover Π is an isomorphism of complete lattices enjoying good properties (see (ii), (iii)).

Let $\mathcal{C}_0(X)$ be the C^*-algebra of continuous functions vanishing at infinity on a locally compact Hausdorff space X, and A its universal representation. The set of all (resp. open, closed) subsets of X is denoted by $\mathcal{P}(X)$ (resp. $\mathcal{G}(X)$, $\mathcal{F}(X)$). The set of $\overline{\mathbb{R}}$-valued (resp. $[-\infty,+\infty[$-valued) upper semicontinuous functions on X is denoted by $\mathcal{USC}(X,\overline{\mathbb{R}})$ (resp. $\mathcal{USC}(X,[-\infty,+\infty[)$), and its $\mathbb{R}$-valued bounded part by $\mathcal{USC}(X,\mathbb{R})_b$.

Let us recall some basic facts. Let M_1 be the set of regular probability measures on X. For each $\mu \in M_1$, we define $\pi_\mu : L^\infty(X,\mu) \to \mathcal{B}(L^2(X,\mu))$ by $(\pi_\mu(f)g)(t) = f(t)g(t)$ (where $\mathcal{B}(L^2(X,\mu))$ denotes the set of bounded linear operators on $L^2(X,\mu)$), and put $\pi_\oplus = \oplus_{\mu\in M_1}\pi_{\mu|\bigcap_{\mu\in M_1} L^\infty(X,\mu)}$. Then, A'' is the strong closure of $\pi_\oplus(\mathcal{C}_0(X))$ in $\mathcal{B}(\oplus_{\mu\in M_1}L^2(X,\mu))$, and $\pi_\oplus(f)$ is q-upper (resp. q-lower) semicontinuous if and only if f is upper (resp. lower) semicontinuous; in particular, a projection $p \in A''$ is closed (resp. open, compact) if and only if $p = \pi_\oplus(1_Y)$ for some closed (resp. open, compact) set $Y \subset X$. Moreover, $E_\lambda^{\pi_\oplus(f)} = \oplus_{\mu\in M_1}E_\lambda^{\pi_\mu(f)}$ for each $f \in \bigcap_{\mu\in M_1} L^\infty(X,\mu)$ and each real λ. The map $\psi^{\mathcal{G}(X)} : \mathcal{USC}(X,\overline{\mathbb{R}}) \to \mathrm{Sup}(\mathcal{G}(X),\overline{\mathbb{R}})$ defined by $\psi^{\mathcal{G}(X)}(f)(G) = \sup_{t\in G} f(t)$ is a bijection with converse map $(\psi^{\mathcal{G}(X)})^{-1}(\gamma)(t) = \inf\{\gamma(G); G \in \mathcal{G}(X), t \in G\}$. Since for all f_1, f_2 in $\mathcal{USC}(X,\overline{\mathbb{R}})$, $f_1 \le f_2$ if and only if $\psi^{\mathcal{G}(X)}(f_1) \le \psi^{\mathcal{G}(X)}(f_2)$, $\psi^{\mathcal{G}(X)}$ is a surjective isomorphism of complete lattices. Let $\phi : \mathrm{Sup}(\mathcal{G}(X),\overline{\mathbb{R}}) \to \mathrm{Sup}(\mathcal{G},\overline{\mathbb{R}})$ be the surjective isomorphism of complete lattices defined by $\phi(\gamma)(\pi_\oplus(1_G)) = \gamma(G)$.

Proposition 4.1. *Let A be commutative with spectrum X.*

(a) *The map $(\psi^{\mathcal{G}})^{-1} \circ \phi \circ \psi^{\mathcal{G}(X)}$ is the unique isomorphism of complete lattices Π' from $\mathcal{USC}(X,\overline{\mathbb{R}})$ into $A''^{\mathcal{G}}_{\overline{\mathbb{R}}}$ which extends $\pi_{\oplus|\mathcal{USC}(X,\mathbb{R})_b}$ and satisfies for each $a \ge 0$, and each $f \in \mathcal{USC}(X,\overline{\mathbb{R}})$ bounded from above,*

$$\Pi'(f \vee -a) = \Pi'(f) \vee -a1, \tag{4.1}$$

and for each $f \in \mathcal{USC}(X,\overline{\mathbb{R}})$ bounded from below,

$$\Pi'(f \wedge a) = \Pi'(f) \wedge a1. \tag{4.2}$$

(b) *For each f and 1_Y in $\bigcap_{\mu\in M_1} L^\infty(X,\mu)$, we have $\psi \circ \pi_\oplus(f)(\pi_\oplus(1_Y)) = \sup_{t\in Y} f(t)$.*

Proof. Let f and 1_Y in $\bigcap_{\mu\in M_1} L^\infty(X,\mu)$, and suppose

$$\lambda_0 = \sup_{t\in Y} f(t) > \sup\{\lambda \in \sigma(\pi_\oplus(f));\ \ \forall\varepsilon > 0,\ \ \pi_\oplus(1_Y)E^{\pi_\oplus(f)}_{]\lambda-\varepsilon,\lambda+\varepsilon]} \neq 0\}.$$

Since $f(t) \in \sigma(\pi_\oplus(f))$ for all $t \in X$, we have $\lambda_0 \in \sigma(\pi_\oplus(f))$, and so there exists $\varepsilon_0 > 0$ such that $\pi_\oplus(1_Y)E^{\pi_\oplus(f)}_{]\lambda_0-\varepsilon_0,\lambda_0+\varepsilon_0]} = 0$. Since $\pi_\oplus(1_Y) \leq E^{\pi_\oplus(f)}_{\lambda_0+\varepsilon}$ for all $\varepsilon > 0$, we have $\pi_\oplus(1_Y) \leq E^{\pi_\oplus(f)}_{\lambda_0-\varepsilon_0}$. This implies $\mu(1_Y) \leq \mu(1_{Y\cap\{f\leq\lambda_0-\varepsilon_0\}})$ for all $\mu \in M_1$, giving a contradiction if we take $\mu = \delta_{t_0}$ the Dirac measure at $t_0 \in Y$, with $\lambda_0 - f(t_0) < \varepsilon_0$. Suppose now

$$\sup_{t\in Y} f(t) < \sup\{\lambda \in \sigma(\pi_\oplus(f)); \quad \forall\varepsilon > 0, \quad \pi_\oplus(1_Y)E^{\pi_\oplus(f)}_{]\lambda-\varepsilon,\lambda+\varepsilon]} \neq 0\}.$$

There exists $\lambda' \in \sigma(\pi_\oplus(f))$ such that $\sup_{t\in Y} f(t) < \lambda'$, and $\pi_\oplus(1_Y) \not\leq E^{\pi_\oplus(f)}_{\lambda'-\varepsilon}$ for all $\varepsilon > 0$. For all $\varepsilon > 0$, there exists $h = \{h_\mu; \mu \in M_1\} \in \oplus_{\mu\in\pi_1} L^2(X, \mu)$ such that $\langle h, \pi_\oplus(1_Y)h\rangle > \langle h, E^{\pi_\oplus(f)}_{\lambda'-\varepsilon} h\rangle$, and so $\langle h_{\mu'}, \pi_\oplus(1_Y)h_{\mu'}\rangle_{L^2(X,\mu')} > \langle h_{\mu'}, E^{\pi_\oplus(f)}_{\lambda'-\varepsilon} h_{\mu'}\rangle_{L^2(X,\mu')}$ for some $\mu' \in M_1$. Equivalently,

$$\mu'(1_Y \mid h_{\mu'} \mid^2) > \mu'(1_{\{f\leq\lambda'-\varepsilon\}} \mid h_{\mu'} \mid^2),$$

which gives a contradiction with ε satisfying $\sup_{t\in Y} f(t) < \lambda' - \varepsilon$. We then have

$$\sup_{t\in Y} f(t) = \sup\{\lambda \in \sigma(\pi_\oplus(f)); \quad \forall\varepsilon > 0, \quad \pi_\oplus(1_Y)E^{\pi_\oplus(f)}_{]\lambda-\varepsilon,\lambda+\varepsilon]} \neq 0\},$$

that is $\sup_{t\in Y} f(t) = \psi\circ\pi_\oplus(f)(\Pi_\oplus(1_Y))$ by Lemma 2.2, and (b) holds. In particular, for each $f \in \mathcal{USC}(X, \mathbb{R})_b$ and each $G \in \mathcal{G}(X)$, we have

$$\phi \circ \psi^{\mathcal{G}(X)}(f)(\pi_\oplus(1_G)) = \sup_{t\in G} f(t) = \psi \circ \pi_\oplus(f)(\pi_\oplus(1_G)) = \psi^{\mathcal{G}} \circ \pi_\oplus(f)(\Pi_\oplus(1_G))$$

(the last equality follows from Theorem 3.3), that is $(\psi^{\mathcal{G}})^{-1} \circ \phi \circ \psi^{\mathcal{G}(X)}(f) = \pi_\oplus(f)$ by Theorem 2.3, hence $(\psi^{\mathcal{G}})^{-1} \circ \phi \circ \psi^{\mathcal{G}(X)}$ extends $\pi_{\oplus|\mathcal{USC}(X,\mathbb{R})_b}$; it is clearly an isomorphism of complete lattices, which moreover satisfies (4.1) and (4.2) since it is surjective with $(\psi^{\mathcal{G}})^{-1} \circ \phi \circ \psi^{\mathcal{G}(X)}(a) = a1$ for each real a. Let $\Pi' : \mathcal{USC}(X, \overline{\mathbb{R}}) \longrightarrow A''^{\mathcal{G}}_{\overline{\mathbb{R}}}$ be an isomorphism of complete lattice (not necessarily surjective) extending $\pi_{\oplus|\mathcal{USC}(X,\mathbb{R})_b}$, and satisfying (4.1) and (4.2). In the rest of the proof, and in absence of explicit mention, the meets and joins of extended q-upper semicontinuous operators are taken in $A''^{\mathcal{G}}_{\overline{\mathbb{R}}}$. Note that for each $g \in \mathcal{USC}(X, \overline{\mathbb{R}})$ bounded from above,

$$\bigwedge_{a\in\mathbb{R}_+} \pi_\oplus(g\vee-a) = \bigwedge_{a\in\mathbb{R}_+} \Pi'(g)\vee-a1 = \Pi'(g) \preceq \bigwedge_{a\in\mathbb{R}_+}^{\Pi'(\mathcal{USC}(X,\overline{\mathbb{R}}))} \Pi'(g\vee-a) \preceq \bigwedge_{a\in\mathbb{R}_+} \Pi'(g)\vee-a1 = \Pi'(g)$$

(the first equality follows from (4.1), and the second one from Corollary 2.4 (c)), hence

$$\bigwedge_{a\in\mathbb{R}_+} \pi_\oplus(g \vee -a) = \bigwedge_{a\in\mathbb{R}_+}^{\Pi'(\mathcal{USC}(X,\overline{\mathbb{R}}))} \Pi'(g \vee -a). \tag{4.3}$$

Note also that for each $G \in \mathcal{G}(X)$,

$$\psi^{\mathcal{G}}(\bigwedge_{a\in\mathbb{R}_+} \pi_\oplus(g\vee-a))(\pi_\oplus(1_G)) = \inf_{a\in\mathbb{R}_+} \psi^{\mathcal{G}}(\pi_\oplus(g\vee-a))(\pi_\oplus(1_G)) = \inf_{a\in\mathbb{R}_+} \sup_{t\in G}(g\vee-a)(t) = \sup_{t\in G} g(t) \tag{4.4}$$

(the proof of the first equality is similar to the proof of Theorem 3.4: suppose $\psi^{\mathcal{G}}(\bigwedge_{a\in\mathbb{R}_+} \pi_\oplus(g \vee -a))(\pi_\oplus(1_G)) < \inf_{a\in\mathbb{R}_+} \psi^{\mathcal{G}}(\pi_\oplus(g \vee -a))(\pi_\oplus(1_G))$, take the family $\{\pi_\oplus(g \vee -a); a \in \mathbb{R}_+\}$ in place of $\{x_i; i \in I\}$, replace p by $\pi_\oplus(1_G)$, and get by commutativity, $\pi_\oplus(1_G).\bigwedge_{a\in\mathbb{R}_+} E^{\pi_\oplus(g\vee-a)}_{[s,+\infty[} = 0$, which implies $\pi_\oplus(1_G) \leq$

$\bigvee_{a\in\mathbb{R}_+} E_s^{\pi_\oplus(g\vee -a)} = E_s^{\pi_\oplus(g\vee s)}$ and the contradiction. The second equality follows from (b) and Theorem 3.3 (i)). Let $f \in \mathcal{USC}(X,\overline{\mathbb{R}})$, and note that $f = \bigvee_{b\in\mathbb{R}_+}\bigwedge_{a\in\mathbb{R}_+}((f\wedge b)\vee -a)$ with $(f\wedge b)$ bounded from above. Since $\psi^{\mathcal{G}}\circ\Pi'$ is sup-preserving as isomorphism of complete lattices, we get by (4.3),

$$\psi^{\mathcal{G}}\circ\Pi'(f) = \bigvee_{b\in\mathbb{R}_+}^{\psi^{\mathcal{G}}\circ\Pi'(\mathcal{USC}(X,\overline{\mathbb{R}}))} \psi^{\mathcal{G}}(\bigwedge_{a\in\mathbb{R}_+}^{\Pi'(\mathcal{USC}(X,\overline{\mathbb{R}}))} \Pi'((f\wedge b)\vee -a))$$

$$= \bigvee_{b\in\mathbb{R}_+}^{\psi^{\mathcal{G}}\circ\Pi'(\mathcal{USC}(X,\overline{\mathbb{R}}))} \psi^{\mathcal{G}}(\bigwedge_{a\in\mathbb{R}_+}\pi_\oplus((f\wedge b)\vee -a)) \geq \bigvee_{b\in\mathbb{R}_+}\psi^{\mathcal{G}}(\bigwedge_{a\in\mathbb{R}_+}\pi_\oplus((f\wedge b)\vee -a)),$$

and by (4.4) for each $G\in\mathcal{G}(X)$,

$$\psi^{\mathcal{G}}\circ\Pi'(f)(\pi_\oplus(1_G)) \geq \sup_{b\in\mathbb{R}_+}\inf_{a\in\mathbb{R}_+}\sup_{t\in G}((f\wedge b)\vee -a)(t) = \sup_{b\in\mathbb{R}_+}\sup_{t\in G}(f\wedge b)(t) \tag{4.5}$$

$$= \sup_{t\in G} f(t) = (\phi\circ\psi^{\mathcal{G}(X)}(f))(\pi_\oplus(1_G)).$$

On the other hand, for each real $a\geq 0$,

$$\psi^{\mathcal{G}}\circ\Pi'(f) \leq \psi^{\mathcal{G}}\circ\Pi'(f\vee -a) \leq \psi^{\mathcal{G}}(\bigvee_{b\in\mathbb{R}_+}\Pi'(f\vee -a)\wedge b1) =$$

$$= \psi^{\mathcal{G}}(\bigvee_{b\in\mathbb{R}_+}\pi_\oplus((f\vee -a)\wedge b)) = \bigvee_{b\in\mathbb{R}_+}\psi^{\mathcal{G}}\circ\pi_\oplus((f\vee -a)\wedge b)$$

(the second inequality follows from Corollary 2.4 (c), and the first equality from (4.2)), which gives for each $G\in\mathcal{G}$,

$$\psi^{\mathcal{G}}\circ\Pi'(f)(\pi_\oplus(1_G)) \leq \inf_{a\in\mathbb{R}_+}\sup_{b\in\mathbb{R}_+}\sup_{t\in G}((f\wedge b)\vee -a)(t) = \sup_{t\in G} f(t) = (\phi\circ\psi^{\mathcal{G}(X)}(f))(\pi_\oplus(1_G)). \tag{4.6}$$

By (4.5) and (4.6), $\Pi' = (\psi^{\mathcal{G}})^{-1}\circ\phi\circ\psi^{\mathcal{G}(X)}$, which proves the uniqueness, and (a) holds. □

The following theorem shows that $\overline{\mathbb{R}}^X$ can be seen as a complete lattice in $A''_{\overline{\mathbb{R}}}$, containing $A''^{\mathcal{G}}_{\overline{\mathbb{R}}}$ as a complete lattice isomorphic to $\mathcal{USC}(X,\overline{\mathbb{R}})$. In this identification, the q-upper regularization coincides with the usual upper regularization. By (4.11) and (4.12), the unbounded q-lower semicontinuous operators in the sense of [14] correspond to the $]-\infty,+\infty]$-valued lower semicontinuous functions on X, and f is $\overline{\mathbb{R}}$-valued (resp. $\mathbb{R}$-valued, bounded $\mathbb{R}$-valued) continuous if and only if $\Pi(f)$ is an extended (resp. extended with $\sigma_{\overline{\mathbb{R}}}(\Pi(f))\subset\mathbb{R}$, bounded) q-continuous operator.

Theorem 4.2. *Let A be commutative with spectrum X. The map $\Pi:\overline{\mathbb{R}}^X\to A''_{\overline{\mathbb{R}}}$ defined by*

$$\Pi(f) = \bigwedge_{a\in\mathbb{R}}(\bigvee_{\lambda-a\in\mathbb{R}_+}(\lambda-a)\bigvee_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)}\pi_\oplus(1_F)+a1) \tag{4.7}$$

satisfies the following properties:

(i) Π *is an isomorphism of complete lattices, and for each* $f \in \overline{\mathbb{R}}^X$ *and each real* μ*, we have*

(4.8)
$$E^{\Pi(f)}_{[\mu,+\infty[} = \bigwedge_{\lambda<\mu} \bigvee_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)} \pi_\oplus(1_F), \quad \text{and} \quad E^{\Pi(f)}_{]\mu,+\infty[} = \bigvee_{\lambda>\mu} \bigvee_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)} \pi_\oplus(1_F).$$

Moreover, for each real a*,*

$$\Pi(f\vee a) = \Pi(f)\vee a1 = \bigvee_{\lambda-a\in\mathbb{R}_+} (\lambda-a) \bigvee_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)} \pi_\oplus(1_F) + a1, \tag{4.9}$$

and

$$\Pi(f\wedge a) = \Pi(f)\wedge a1. \tag{4.10}$$

(ii) $\Pi_{|\bigcap_{\mu\in\mathcal{M}_1} L^\infty(X,\mu)} = \pi_\oplus$.

(iii) $\Pi_{|\mathcal{USC}(X,\overline{\mathbb{R}})} = (\psi^{\mathcal{G}})^{-1}\circ\phi\circ\psi^{\mathcal{G}(X)}$. *In particular,* $\Pi_{|\mathcal{USC}(X,\overline{\mathbb{R}})}$ *is an isomorphism of complete lattices onto* $A''^{\mathcal{G}}_{\overline{\mathbb{R}}}$*. Moreover,*

$$-f\in\mathcal{USC}(X,\overline{\mathbb{R}}) \iff -\Pi(f)\in A''^{\mathcal{G}}_{\overline{\mathbb{R}}} \tag{4.11}$$

and

$$f\in\mathcal{USC}(X,[-\infty,+\infty[) \iff \Pi(f)\in A''^{\mathcal{G}}_{[-\infty,+\infty[}. \tag{4.12}$$

(iv) $\overline{\Pi(f)} = \Pi(\overline{f})$ *for all* $f\in\overline{\mathbb{R}}^X$.

(v) $\psi\circ\Pi(f)(\pi_\oplus(1_G)) = \sup_{t\in G} f(t)$ *for all* $f\in\overline{\mathbb{R}}^X$ *and all* $G\in\mathcal{G}(X)$.

Proof. We first show that for each $f\in\overline{\mathbb{R}}^X$ and each real a,

$$\Pi(f\vee a) = \bigvee_{\lambda-a\in\mathbb{R}_+} (\lambda-a) \bigvee_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)} \pi_\oplus(1_F) + a1. \tag{4.13}$$

Let $a\in\mathbb{R}$, and suppose that for some real $a'\geq a$,

$$\psi(\bigvee_{\lambda-a'\in\mathbb{R}_+} (\lambda-a') \bigvee_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)} \pi_\oplus(1_F)+a'1) \not\succeq \psi(\bigvee_{\lambda-a\in\mathbb{R}_+} (\lambda-a) \bigvee_{F\subset\{f\vee a\geq\lambda\},F\in\mathcal{F}(X)} \pi_\oplus(1_F)+a1).$$

By the sup-preserving property of ψ, and using Lemma 2.2 (e), we get a projection $p\in A''\backslash\{0\}$, a real $\lambda_0\geq a$, and a closed set $F_0\subset\{f\vee a\geq\lambda_0\}$ such that

(4.14)
$$\sup_{\lambda-a'\in\mathbb{R}_+} (\lambda-a') \bigvee_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)} \psi(\pi_\oplus(1_F))(p) < (\lambda_0-a)\psi(\pi_\oplus(1_{F_0}))(p)+a-a'.$$

Since $\psi(\pi_\oplus(1_F))(p)\in\{0,1\}$ for any $F\in\mathcal{F}(X)$ by Lemma 2.2 (c), the L.H.S. in (4.14) is positive, which implies $\psi(\pi_\oplus(1_{F_0}))(p) = 1$ and $\lambda_0 > a' \geq a$. Therefore, $F_0\subset\{f\geq\lambda_0\}$, and by taking $\lambda=\lambda_0$ and $F=F_0$ in (4.14) we get the contradiction. By Theorem 2.3, we then have for each real $a'\geq a$,

(4.15)
$$\bigvee_{\lambda-a'\in\mathbb{R}_+} (\lambda-a') \bigvee_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)} \pi_\oplus(1_F)+a'1 \succeq \bigvee_{\lambda-a\in\mathbb{R}_+} (\lambda-a) \bigvee_{F\subset\{f\vee a\geq\lambda\},F\in\mathcal{F}(X)} \pi_\oplus(1_F)+a1,$$

hence

$$\bigvee_{\lambda-a'\in\mathbb{R}_+} (\lambda-a') \bigvee_{F\subset\{f\vee a\geq\lambda\},F\in\mathcal{F}(X)} \pi_\oplus(1_F)+a'1 \succeq \bigvee_{\lambda-a'\in\mathbb{R}_+} (\lambda-a') \bigvee_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)} \pi_\oplus(1_F)+a'1$$

$$\succeq \bigvee_{\lambda-a\in\mathbb{R}_+}(\lambda-a)\bigvee_{F\subset\{f\vee a\geq\lambda\},F\in\mathcal{F}(X)}\pi_\oplus(1_F)+a1 \succeq \bigvee_{\lambda-a\in\mathbb{R}_+}(\lambda-a)\bigvee_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)}\pi_\oplus(1_F)+a1$$

$$\succeq \bigvee_{\lambda-a\in\mathbb{R}_+}(\lambda-a)\bigvee_{F\subset\{f\vee a\geq\lambda\},F\in\mathcal{F}(X)}\pi_\oplus(1_F)+a1 \tag{4.16}$$

(where the second inequality follows from (4.15), and the last one from (4.15) with $a'=a$), and so

$$\bigwedge_{a'\geq a}\Big(\bigvee_{\lambda-a'\in\mathbb{R}_+}(\lambda-a')\bigvee_{F\subset\{f\vee a\geq\lambda\},F\in\mathcal{F}(X)}\pi_\oplus(1_F)+a'1\Big)=\bigvee_{\lambda-a\in\mathbb{R}_+}(\lambda-a)\bigvee_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)}\pi_\oplus(1_F)+a1. \tag{4.17}$$

On the other hand, for each real $a'\leq a$, we have

$$\bigvee_{\lambda-a'\in\mathbb{R}_+}(\lambda-a')\bigvee_{F\subset\{f\vee a\geq\lambda\},F\in\mathcal{F}(X)}\pi_\oplus(1_F)+a'1=\Big(\bigvee_{a'\leq\lambda\leq a}(\lambda-a')\bigvee_{F\subset\{f\vee a\geq\lambda\},F\in\mathcal{F}(X)}\pi_\oplus(1_F)+a'1\Big)\vee \tag{4.18}$$

$$\Big(\bigvee_{\lambda>a}(\lambda-a')\bigvee_{F\subset\{f\vee a\geq\lambda\},F\in\mathcal{F}(X)}\pi_\oplus(1_F)+a'1\Big)=a1\vee\Big(\bigvee_{\lambda>a}(\lambda-a')\bigvee_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)}\pi_\oplus(1_F)+a'1\Big)\succeq$$

$$a1\vee\Big(\bigvee_{\lambda>a}(\lambda-a)\bigvee_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)}\pi_\oplus(1_F)+a1\Big)=\bigvee_{\lambda-a\in\mathbb{R}_+}(\lambda-a)\bigvee_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)}\pi_\oplus(1_F)+a1,$$

where the second equality follows by noting that $\{f\vee a\geq\lambda\}=X$ for $a\geq\lambda$ and $\{f\vee a\geq\lambda\}=\{f\geq\lambda\}$ for $\lambda>a$. Then, (4.13) follows from (4.17) and (4.18). By (4.13) we get $E_\mu^{\Pi(f\vee a)}=\bigwedge_{\lambda>a}\bigwedge_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)}E^{\pi_\oplus(1_F)}_{\frac{\mu-a}{\lambda-a}}\wedge E^0_{\mu-a}$ for each real μ, so that

$$\forall\mu<a,\qquad E_\mu^{\Pi(f\vee a)}=0. \tag{4.19}$$

On the other hand,

$$E_\mu^{\Pi(f)}=\bigwedge_{\nu>\mu}\bigvee_{a\in\mathbb{R}}\bigwedge_{\lambda>a}\bigwedge_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)}E^{\pi_\oplus(1_F)}_{\frac{\nu-a}{\lambda-a}}\wedge E^0_{\nu-a}=\bigwedge_{\nu>\mu}\bigvee_{a\leq\nu}\bigwedge_{\lambda>a}\bigwedge_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)}E^{\pi_\oplus(1_F)}_{\frac{\nu-a}{\lambda-a}}\wedge E^0_{\nu-a},$$

and since $\bigwedge_{\lambda>a}\bigwedge_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)}E^{\pi_\oplus(1_F)}_{\frac{\nu-a}{\lambda-a}}\wedge E^0_{\nu-a}$ is constant for all reals a and ν with $a\leq\nu$, we obtain

$$\forall\mu\geq a,\qquad E_\mu^{\Pi(f)}=\bigwedge_{\lambda>a}\bigwedge_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)}E^{\pi_\oplus(1_F)}_{\frac{\mu-a}{\lambda-a}}=E_\mu^{\Pi(f\vee a)}. \tag{4.20}$$

Since $E_\mu^{\Pi(f)\vee a1}=E_\mu^{\Pi(f)}$ for $\mu\geq a$ and $E_\mu^{\Pi(f)\vee a1}=0$ for $\mu<a$, the first equality in (4.9) follows from (4.19) and (4.20), and the second one from (4.13). Take $\mu=a$ in (4.20) and get

$$E_\mu^{\Pi(f)}=\bigwedge_{\lambda>\mu}\bigwedge_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)}1-\pi_\oplus(1_F), \tag{4.21}$$

which is equivalent to the second equality in (4.8). By (4.21) we have

$$E^{\Pi(f)}_{]-\infty,\nu[}=\bigvee_{\nu'<\nu}\Big(\bigwedge_{\nu'<\lambda<\nu}\bigwedge_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)}1-\pi_\oplus(1_F)\Big)\leq\bigwedge_{F\subset\{f\geq\nu\},F\in\mathcal{F}(X)}1-\pi_\oplus(1_F)\leq E_\nu^{\Pi(f)},$$

so that

$$E^{\Pi(f)}_{]-\infty,\mu[}=\bigvee_{\nu<\mu}E^{\Pi(f)}_{]-\infty,\nu[}\leq\bigvee_{\nu<\mu}\bigwedge_{F\subset\{f\geq\nu\},F\in\mathcal{F}(X)}1-\pi_\oplus(1_F)\leq\bigvee_{\nu<\mu}E_\nu^{\Pi(f)}=E^{\Pi(f)}_{]-\infty,\mu[},$$

which is equivalent to the first equality in (4.8). By (4.21) we get

$$\forall \mu \geq a, \qquad E_\mu^{\Pi(f\wedge a)} = 1 \tag{4.22}$$

and

$$\forall \mu < a, \qquad E_\mu^{\Pi(f\wedge a)} = \bigwedge_{\lambda>\mu} \bigwedge_{F\subset\{f\wedge a\geq\lambda\},F\in\mathcal{F}(X)} 1 - \pi_\oplus(1_F) =$$

$$\bigwedge_{a>\lambda>\mu} \bigwedge_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)} 1 - \pi_\oplus(1_F) = E_\mu^{\Pi(f)}. \tag{4.23}$$

Since $E_\mu^{\Pi(f)\wedge a1} = E_\mu^{\Pi(f)}$ for $\mu < a$ and $E_\mu^{\Pi(f)\wedge a1} = 1$ for $\mu \geq a$, (4.10) follows from (4.22) and (4.23). Let f and g in $\overline{\mathbb{R}}^X$. Suppose $\Pi(f) \preceq \Pi(g)$ and $f(x) > \lambda_0 > g(x)$ for some $x \in X$ and some real λ_0. Then, $\Pi(f \vee \lambda_0) - \lambda_0 1 \preceq \Pi(g \vee \lambda_0) - \lambda_0 1$ by (4.9), which implies $\psi(\Pi(f \vee \lambda_0) - \lambda_0 1) \leq \psi(\Pi(g \vee \lambda_0) - \lambda_0 1)$ by Theorem 2.3, and

$$\psi(\bigvee_{\lambda-\lambda_0\in\mathbb{R}_+} (\lambda-\lambda_0) \bigvee_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)} \pi_\oplus(1_F)) \leq \psi(\bigvee_{\lambda-\lambda_0\in\mathbb{R}_+} (\lambda-\lambda_0) \bigvee_{F\subset\{g\geq\lambda\},F\in\mathcal{F}(X)} \pi_\oplus(1_F))$$

by (4.13) and Lemma 2.2 (e), that is for each projection $p \in A''$,

$$\sup_{\lambda-\lambda_0\in\mathbb{R}_+} (\lambda-\lambda_0) \sup_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)} \psi(\pi_\oplus(1_F))(p) \leq \sup_{\lambda-\lambda_0\in\mathbb{R}_+} (\lambda-\lambda_0) \sup_{F\subset\{g\geq\lambda\},F\in\mathcal{F}(X)} \psi(\pi_\oplus(1_F))(p). \tag{4.24}$$

Take $p = \pi_\oplus(1_{\{x\}})$ in (4.24), and get the contradiction by considering in the L.H.S. any $\lambda \in]\lambda_0, f(x)[$ and $\{x\} \subset \{f \geq \lambda\}$ (indeed, $\psi(p)(p) = 1$ gives $\lambda - \lambda_0 > 0$ in the L.H.S., but $\psi(\pi_\oplus(1_F))(p) = \sup_{t\in X} 1_{F\cap\{x\}}(t) = 0$ for all $F \subset \{g \geq \lambda\}$ and all $\lambda \geq \lambda_0$ by Proposition 4.1, which gives 0 in the R.H.S.). Therefore, $\Pi(f) \preceq \Pi(g)$ implies $f \leq g$, and since the converse implication clearly holds, Π is an isomorphism of complete lattices and (i) holds.

Since $\pi_\oplus(1_{\{f\geq\lambda\}}) = E_{[\lambda,+\infty[}^{\pi_\oplus(f)}$ for each $f \in \bigcap_{\mu\in M_1} L^\infty(X,\mu)$, we have for each $f \in \mathcal{USC}(X,\mathbb{R})_b$ and each real $a \leq \inf_{t\in X} f(t)$,

$$\Pi(f) = \bigvee_{\lambda-a\in\mathbb{R}_+} (\lambda-a) \bigvee_{F\subset\{f\geq\lambda\},F\in\mathcal{F}(X)} \pi_\oplus(1_F) + a1 = \bigvee_{\lambda-a\in\mathbb{R}_+} (\lambda-a) E_{[\lambda,+\infty[}^{\pi_\oplus(f)} + a1 = \pi_\oplus(f),$$

where the first equality follows from (4.13), and the last one from Proposition 4.1 by noting that $f = \bigvee_{\lambda-a\in\mathbb{R}_+} (\lambda - a) 1_{\{f\geq\lambda\}} + a$; this proves (ii).

Let $f \in \mathcal{USC}(X, \overline{\mathbb{R}})$. For each real b, we have

$$\Pi(f\wedge b) = \Pi(f)\wedge b1 = \bigwedge_{a\in\mathbb{R}} (\Pi(f)\wedge b1)\vee a1) = \bigwedge_{a\in\mathbb{R}} \Pi((f\wedge b)\vee a)) = \bigwedge_{a\in\mathbb{R}} \pi_\oplus((f\wedge b)\vee a), \tag{4.25}$$

where the first equality follows from (4.10), the second one from Corollary 2.4 (c), the third one from (4.9) and (4.10), and the last one from (ii) since $(f \wedge b) \vee a \in \mathcal{USC}(X,\mathbb{R})_b$. Therefore, $\Pi(f \wedge b) \in A''^{\mathcal{G}}_{\overline{\mathbb{R}}}$ as the meet in $A''_{\overline{\mathbb{R}}}$ of a family of q-upper semicontinuous operators. Since $\Pi(f) = \bigvee_{b\in\mathbb{R}} \Pi(f \wedge b)$ by (4.10) and Corollary 2.4 (c), and since

$$\forall \lambda \in \mathbb{R}, \qquad E_{]-\infty,\lambda[}^{\Pi(f)} = \bigvee_{\nu<\lambda} E_\nu^{\Pi(f)} = \bigvee_{\nu<\lambda} E_\nu^{\Pi(f\wedge\lambda)} = \bigvee_{\nu<\lambda} E_{]-\infty,\nu[}^{\Pi(f\wedge\lambda)},$$

$E_{]-\infty,\lambda[}^{\Pi(f)}$ is an open projection, hence $\Pi(f) \in A''^{\mathcal{G}}_{\overline{\mathbb{R}}}$. We then have shown that $\Pi_{|\mathcal{USC}(X,\overline{\mathbb{R}})}$ is an isomorphism of complete lattices from $\mathcal{USC}(X, \overline{\mathbb{R}})$ into $A''^{\mathcal{G}}_{\overline{\mathbb{R}}}$, which extends $\pi_{\oplus|\mathcal{USC}(X,\mathbb{R})_b}$, and satisfies (4.1) and (4.2) ; by Proposition 4.1, $\Pi_{|\mathcal{USC}(X,\overline{\mathbb{R}})} = (\psi^{\mathcal{G}})^{-1} \circ \phi \circ \psi^{\mathcal{G}(X)}$ and the two first assertions of (iii) holds. Since for each real a,

$$\bigvee_{\lambda\in\mathbb{R}} E_\lambda^{\Pi(f)} = \bigvee_{\lambda\in\mathbb{R}} E_\lambda^{\Pi(f\vee a)} = \bigvee_{\lambda\in\mathbb{R}} E_\lambda^{\Pi(f\vee a\wedge\lambda)} = \bigvee_{\lambda\in\mathbb{R}} E_\lambda^{\pi_\oplus(f\vee a\wedge\lambda)} = \bigvee_{\lambda\in\mathbb{R}} \Pi(1_{\{f\vee a\leq\lambda\}}) = \tag{4.26}$$

$$= \bigvee_{\lambda\in\mathbb{R}} \Pi(1_{\{f\leq\lambda\}}) \leq \Pi(1_{\bigcup_{\lambda\in\mathbb{R}}\{f\leq\lambda\}}) \leq \Pi(1_X) = 1,$$

$\Pi(f) \in A''^{\mathcal{G}}_{[-\infty,+\infty[}$ implies $\bigcup_{\lambda\in\mathbb{R}}\{f \leq \lambda\} = X$; but the first inequality in 4.26 is in fact an equality (otherwise, by Theorem 2.3, there is some real λ_0 and some projection $p \in A''\backslash\{0\}$ satisfying $(\psi\circ\Pi)(1_{\{f\leq\lambda_0\}})(p) > (\psi\circ\Pi)(1_{\bigcup_{\lambda\in\mathbb{R}}\{f\leq\lambda\}})(p)$; since $\Pi(1_{\{f\leq\lambda_0\}})$ and $\Pi(1_{\bigcup_{\lambda\in\mathbb{R}}\{f\leq\lambda\}})$ are projections by (ii), we get $(\psi\circ\Pi)(1_{\{f\leq\lambda_0\}})(p) = 1$ that is $p \leq \Pi(1_{\{f\leq\lambda_0\}})$, and $(\psi\circ\Pi)(1_{\bigcup_{\lambda\in\mathbb{R}}\{f\leq\lambda\}})(p) = 0$ that is $p.\Pi(1_{\bigcup_{\lambda\in\mathbb{R}}\{f\leq\lambda\}}) = 0$, and the contradiction), so that the converse implication holds, which gives (4.12). Let $-f \in \mathcal{USC}(X,\overline{\mathbb{R}})$. By (4.9), (4.10), (ii) and Corollary 2.4, we get

$$-\Pi(f) = \bigwedge_{b\in\mathbb{R}} \bigvee_{a\in\mathbb{R}} -\Pi((f\wedge b)\vee a) = \bigwedge_{b\in\mathbb{R}} \bigvee_{a\in\mathbb{R}} \Pi((-f\wedge -a)\vee -b) = \Pi(-f),$$

which gives (4.11).

Let $f \in \overline{\mathbb{R}}^X$. Clearly, $\underline{\Pi(f)} \preceq \overline{\Pi(f)} \preceq \Pi(\overline{f})$, and by (i) and (iii) there exists $g \in \mathcal{USC}(X,\overline{\mathbb{R}})$ such that $\overline{\Pi(f)} = \Pi(g)$ and $f \leq g \leq \overline{f}$, hence $g = \overline{f}$, and (iv) holds.

We have for each $f \in \overline{\mathbb{R}}^X$ and each $G \in \mathcal{G}(X)$,

$$\psi\circ\Pi(f)(\pi_\oplus(1_G)) = \psi\circ\overline{\Pi(f)}(\pi_\oplus(1_G)) = \psi^{\mathcal{G}}\circ\overline{\Pi(f)}(\pi_\oplus(1_G)) = \psi^{\mathcal{G}}\circ\Pi(\overline{f})(\pi_\oplus(1_G)) = \sup_{t\in G} f(t),$$

where the two first equalities follow from Theorem 3.3, the third one from (iv), and the last one from (iii). Thus, (v) holds and the theorem is proved. □

We recover in the following corollary the fact that the spectrum X of a commutative von Neumann algebra is stonean, since this is equivalent to say that the set of $\overline{\mathbb{R}}$-valued continuous functions on X is a complete lattice.

Corollary 4.3. *Let A be a commutative W^*-algebra with spectrum X. For each faithful representation $\pi(A)$ as a von Neumann algebra, there is an isomorphism of complete lattices from the set of $\overline{\mathbb{R}}$-valued continuous functions on X onto $\pi(A)_{\overline{\mathbb{R}}}$.*

Proof. Since π preserves the usual order, by commutativity π restricted to the self-adjoint part of A is a poset isomorphism (with respect to the spectral order). We then can identify (as lattices) the self-adjoint part of A with the one of $\pi(A)$. On the other hand, also by commutativity, A is a von Neumann algebra acting on the universal Hilbert space. By definition, $x \in A_{\overline{\mathbb{R}}}$ if $\{E_\lambda^x; \lambda \in \mathbb{R}\} \subset A$, that is $x \in A_{\overline{\mathbb{R}}}$ if and only if $x\wedge b1\vee -a1 \in A$ for all reals a and b, i.e. x is extended q-continuous, and the result follows from Theorem 4.2 (note that in general, x is extended q-upper (resp. q-lower) semicontinuous if and only if $(x\wedge b1)\vee a1$ is bounded q-upper (resp. q-lower) semicontinuous for all reals a and b). □

The following characterizations $(i') - (iii')$ of the upper regularization map are well known ([8], [16]). When A is commutative, they are recovered as particular cases of Theorem 3.3.

Corollary 4.4. *Let A be commutative with spectrum X, $f \in \overline{\mathbb{R}}^X$ and $Y \subset X$. The properties $(i)-(iii)$ of Theorem 3.3 with $x = \Pi(f)$ and $p = \Pi(1_Y)$ are respectively equivalent to*

(i') $\overline{f}$ *is the unique* $g \in \mathcal{USC}(X, \overline{\mathbb{R}})$ *satisfying*

$$\sup_{t\in G} g(t) = \sup_{t\in G} f(t) \qquad for\ all\ G \in \mathcal{G}(X); \tag{4.27}$$

(ii') $\overline{f} = \bigvee\{g \in \overline{\mathbb{R}}^X; \sup_{t\in G} g(t) = \sup_{t\in G} f(t)\ for\ all\ G \in \mathcal{G}(X)\}$;

(iii') $\psi(\Pi(\overline{f}))(\Pi(1_Y)) = \sup_{t\in Y} \overline{f}(t) = \inf_{G\supset Y, G\in\mathcal{G}(X)} \sup_{t\in G} f(t)$.

Proof. By (iii) and (iv) of Theorem 4.2, (i) is equivalent to state that $\Pi(\overline{f})$ is the unique extended q-upper semicontinuous operator $\Pi(g)$ (for some $g \in \mathcal{USC}(X, \overline{\mathbb{R}})$) satisfying

$$\psi^{\mathcal{G}}(\Pi(g))(\pi_\oplus(1_G)) = \psi(\Pi(f))(\pi_\oplus(1_G)). \tag{4.28}$$

The L.H.S. and R.H.S. of (4.28) are respectively equal to $\sup_{t\in G} g(t)$ by Theorem 4.2 (iii), and $\sup_{t\in G} g(t)$ by Theorem 4.2 (v), which gives (4.27). Put $Y_f = \{g \in \overline{\mathbb{R}}^X; \forall G \in \mathcal{G}(X),\ \sup_{t\in G} g(t) = \sup_{t\in G} f(t)\}$ and $Y_{\Pi(f)} = \{y \in A''_{\overline{\mathbb{R}}}; \psi(y)_{|\mathcal{G}} = \psi(x)_{|\mathcal{G}}\}$. Since $y \in Y_{\Pi(f)}$ if and only if $\overline{y} \in Y_{\Pi(f)}$ by Theorem 3.3 (i), we have by Theorem 4.2 (iii), $\bigvee Y_{\Pi(f)} = \bigvee(Y_{\Pi(f)} \cap \Pi(\overline{\mathbb{R}}^X))$. Moreover, $\overline{\bigvee Y_{\Pi(f)}} \in Y_{\Pi(f)} \cap \Pi(\overline{\mathbb{R}}^X)$ since $\bigvee Y_{\Pi(f)} \in Y_{\Pi(f)}$, hence

$$\bigvee Y_{\Pi(f)} = \overline{\bigvee Y_{\Pi(f)}} = \bigvee^{\Pi(\overline{\mathbb{R}}^X)} (Y_{\Pi(f)} \cap \Pi(\overline{\mathbb{R}}^X)). \tag{4.29}$$

Then,

$$\Pi(\overline{f}) = \overline{\Pi(f)} = \bigvee Y_{\Pi(f)} = \bigvee^{\Pi(\overline{\mathbb{R}}^X)} \Pi(Y_f) = \Pi(\bigvee Y_f), \tag{4.30}$$

where the first equality follows from Theorem 4.2 (iv), the third one from (4.29) and the equivalence $\Pi(g) \in Y_{\Pi(f)}$ if and only if $g \in Y_f$ (given by Theorem 4.2 (v)), and the last one by Theorem 4.2 (i). Then, (4.30) is equivalent to $\overline{f} = \bigvee Y_f$ by Theorem 4.2 (i), that is $(ii) \Leftrightarrow (ii')$. Note that

$$\Pi(\overline{f}) = \Pi(\bigvee_{b\in\mathbb{R}} \bigwedge_{a\in\mathbb{R}} (\overline{f} \wedge b) \vee a)) = \bigvee_{b\in\mathbb{R}}^{\Pi(\mathcal{USC}(X,\overline{\mathbb{R}}))} \Pi(\bigwedge_{a\in\mathbb{R}} (\overline{f} \wedge b) \vee a) \geq$$

$$\bigvee_{b\in\mathbb{R}} \Pi(\bigwedge_{a\in\mathbb{R}} (\overline{f} \wedge b) \vee a) = \bigvee_{b\in\mathbb{R}} \bigwedge_{a\in\mathbb{R}} \Pi((\overline{f} \wedge b) \vee a) = \bigvee_{b\in\mathbb{R}} \bigwedge_{a\in\mathbb{R}} (\Pi(\overline{f}) \wedge b1) \vee a1 = \Pi(\overline{f})$$

(the third equality follows from Proposition 3.2, the fourth one from Theorem 4.2, and the last one from Corollary 2.4); in particular,

$$\bigvee_{b\in\mathbb{R}}^{\Pi(\mathcal{USC}(X,\overline{\mathbb{R}}))} \Pi(\bigwedge_{a\in\mathbb{R}} (\overline{f} \wedge b) \vee a) = \bigvee_{b\in\mathbb{R}} \bigwedge_{a\in\mathbb{R}} \Pi((\overline{f} \wedge b) \vee a). \tag{4.31}$$

Moreover, for each $F \in \mathcal{F}(X)$, we have

$$\psi(\bigwedge_{a\in\mathbb{R}} \Pi((\overline{f}\wedge b)\vee a))(\pi_\oplus(1_F)) = \inf_{a\in\mathbb{R}} \psi\circ\Pi((\overline{f}\wedge b)\vee a)(\pi_\oplus(1_F)) = \inf_{a\in\mathbb{R}} \sup_{t\in F}((\overline{f}\wedge b)\vee a)(t) = \sup_{t\in F} \overline{f}(t)\wedge b, \tag{4.32}$$

where the first equality is proved as the first equality in (4.4), and the second one follows from Proposition 4.1 (b). Now, the L.H.S. of (iii) with $p' = \bigvee_{F\subset Y, F\in\mathcal{F}(X)} \pi_\oplus(1_F)$ is

$$\psi(\overline{\Pi(f)})(p') = \psi \circ \Pi(\overline{f})(p') = \sup_{F\subset Y, F\in\mathcal{F}(X)} \psi \circ \Pi(\overline{f})(\pi_\oplus(1_F)) =$$

$$\sup_{F\subset Y, F\in\mathcal{F}(X)} \psi\circ\Pi(\bigvee_{b\in\mathbb{R}}\bigwedge_{a\in\mathbb{R}}(\overline{f}\wedge b)\vee a)(\pi_\oplus(1_F)) = \sup_{F\subset Y, F\in\mathcal{F}(X)} \psi(\bigvee_{b\in\mathbb{R}}^{\Pi(\mathcal{USC}(X,\overline{\mathbb{R}}))} \Pi(\bigwedge_{a\in\mathbb{R}}(\overline{f}\wedge b)\vee a))(\pi_\oplus(1_F))$$

$$= \sup_{F\subset Y, F\in\mathcal{F}(X)} \psi(\bigvee_{b\in\mathbb{R}}\bigwedge_{a\in\mathbb{R}} \Pi((\overline{f}\wedge b)\vee a))(\pi_\oplus(1_F)) = \sup_{F\subset Y, F\in\mathcal{F}(X)} \sup_{b\in\mathbb{R}} \psi(\bigwedge_{a\in\mathbb{R}} \Pi((\overline{f}\wedge b)\vee a))(\pi_\oplus(1_F)) =$$

$$\sup_{t\in Y} \overline{f}(t),$$

where the fifth equality follows from (4.31), and the last one from (4.32). The R.H.S. of (iii) with p' is

$$\inf_{q\geq p', q\in\mathcal{G}} \psi \circ \Pi(f)(q) = \inf_{\pi_\oplus(1_G)\geq p', G\in\mathcal{G}(X)} \psi \circ \Pi(f)(\pi_\oplus(1_G))$$

$$= \inf_{G\supset Y, G\in\mathcal{G}(X)} \psi \circ \Pi(f)(\pi_\oplus(1_G)) = \inf_{G\supset Y, G\in\mathcal{G}(X)} \sup_{t\in G} f(t) = \inf_{G\supset Y, G\in\mathcal{G}(X)} \sup_{t\in G} \overline{f}(t)$$

$$= \inf_{G\supset Y, G\in\mathcal{G}(X)} \psi \circ \Pi(\overline{f})(\pi_\oplus(1_G)) = \inf_{q\geq\Pi(1_Y), G\in\mathcal{G}} \psi \circ \Pi(\overline{f})(q) = \psi(\overline{\Pi(f)})(\Pi(1_Y)),$$

where the third and fifth equalities follow from Theorem 4.2 (v), and the fourth one from (i'). Since the last equality is exactly (iii) with $p = \Pi(1_Y)$, all the expressions appearing in (iii) and (iii') are equal. □

The commutative case of Theorem 3.4 is exactly the Dini-Cartan lemma, as shows the following.

Corollary 4.5. *Let A be commutative with spectrum X, and $(f_i)_{i\in I}$ be a decreasing net in $\mathcal{USC}(X, \overline{\mathbb{R}})$. The conclusion of Theorem 3.4 with $x_i = \Pi(f_i)$ for all $i \in I$, is equivalent to*

$$\sup_{t\in K}\inf_{i\in I} f_i(t) = \inf_{i\in I}\sup_{t\in K} f_i(t) \qquad \text{for all compact } K \subset X. \tag{4.33}$$

Proof. Note first that for each $i \in I$, $\Pi(f_i)$ is extended q-upper semicontinuous by Theorem 4.2. By commutativity, we have to show that for each compact set $K \subset X$, (3.3) with $p = \pi_\oplus(1_K)$ is equivalent to (4.33). The L.H.S. of (3.3) with $p = \pi_\oplus(1_K)$ is

$$\psi(\bigwedge_{i\in I} \Pi(f_i))(\pi_\oplus(1_K)) = \psi(\bigwedge_{i\in I}^{A''^{\mathcal{G}}_{\mathbb{R}}} \Pi(f_i))(\pi_\oplus(1_K)) = \psi(\bigwedge_{i\in I}^{\Pi(\mathcal{USC}(X,\overline{\mathbb{R}}))} \Pi(f_i))(\pi_\oplus(1_K))$$

$$= \psi(\Pi(\bigwedge_{i\in I} f_i))(\pi_\oplus(1_K)) = \sup_{t\in K}\inf_{i\in I} f_i(t),$$

where the first equality follows from Proposition 3.2, the second and third ones from Theorem 4.2 (iii), and the last one from Corollary 4.4 (iii'). By Corollary 4.4 (iii'), the R.H.S. of (3.3) with $p = \pi_\oplus(1_K)$ is $\inf_{i\in I} \psi(\Pi(f_i))(\pi_\oplus(1_K)) = \inf_{i\in I} \sup_{t\in K} f_i(t)$. □

5. An application to noncommutative large deviations

Let (μ_α) be a net of Radon probability measures on a locally compact Hausdorff space X, let (t_α) be a net in $]0,+\infty[$ converging to 0. Recall that (μ_α) satisfies a large deviation principle with powers (t_α) if there exists a upper semicontinuous function f on X such that

$$\limsup \mu_\alpha(F)^{t_\alpha} \le \sup_{x\in F} f(x) \le \sup_{x\in G} f(x) \le \liminf \mu_\alpha(G)^{t_\alpha}$$

for all $F \in \mathcal{F}(X)$, $G \in \mathcal{G}(X)$ with $F \subset G$ ([8], [10]). Such a principle implies that for each $x \in X$ with $f(x) < 1$, there exists an open set G containing x such that $\mu_\alpha(G)$ converges exponentially fast to 0, with the rate of convergence controlled by $-\log f$; we call f the governing function.

Let (ω_α) be a net of states on a C^*-algebra A. The following definition is a noncommutative version of the above one: indeed, if A is commutative with spectrum X, then (ω_α) is a net of Radon probability measures on X, $z = \pi_\oplus(f)$ for some bounded upper semicontinuous function f, and for each $p = \pi_\oplus(1_Y)$ with Y open or closed, we have by Proposition 4.1 (b),

$$\psi(z)(p) = \psi \circ \pi_\oplus(f)(\pi_\oplus(1_Y)) = \sup_{x\in Y} f(x).$$

Definition 5.1. We say that (ω_α) satisfies a *large deviation principle with powers* (t_α) if there exists a q-upper semicontinuous operator z (called the *governing* operator) such that

$$\limsup \omega_\alpha(p)^{t_\alpha} \le \psi(z)(p) \le \psi(z)(q) \le \liminf \omega_\alpha(q)^{t_\alpha} \tag{5.1}$$

for all closed projections p, and all open projections q with $p \le q$.

We refer to [7], where basic properties of classical large deviations have been extended to the noncommutative context; in particular, we proved that a governing operator is uniquely determined.

The results of the preceding sections allow us to obtain equivalent definitions without reference to semicontinuity; the following proposition is a noncommutative version of Proposition 2.3 of [8].

Proposition 5.2. *The following statements are equivalent:*

(i) (ω_α) *satisfies a large deviation principle with powers* (t_α).

(ii) *There exists a sup-preserving map* γ *on the set of open projections such that for each closed projection* p, *and each open projection* q *with* $p \le q$,

$$\limsup \omega_\alpha(p)^{t_\alpha} \le \gamma(q) \le \liminf \omega_\alpha(q)^{t_\alpha}.$$

(iii) *There exists* $x \in A''_{[0,1]}$ *such that for each closed projection* p, *and each open projection* q *with* $p \le q$,

$$\limsup \omega_\alpha(p)^{t_\alpha} \le \psi(x)(q) \le \liminf \omega_\alpha(q)^{t_\alpha}. \tag{5.2}$$

If (i) *holds, then the governing operator* z *is given by* $z = \bigvee\{x \in A''_{[0,1]} : x \; satisfies \; (5.2)\}$. *If* (ii) *holds, then* (i) *holds with governing operator* $(\psi^{\mathcal{G}})^{-1}(\gamma)$. *If* (iii) *holds with* x, *then* (i) *holds with governing operator* $\overline{x}$.

Proof. $(i) \Rightarrow (ii) \Rightarrow (iii)$ is clear by Theorem 2.3 and Theorem 3.3 (i) and (iii). If (iii) holds with x, then $\psi^{\mathcal{G}}(\overline{x})(q) = \psi(\overline{x})(q) = \psi(x)(q)$ for all $q \in \mathcal{G}$ by Theorem 3.3 (i) and (iii), and since $\psi(\overline{x})(p) = \inf_{q\ge p, q\in\mathcal{G}} \psi(x)(q)$ by Theorem 3.3 (iii), we get (5.1) with $z = \overline{x}$, i.e., (i) holds with governing operator $\overline{x}$. The equivalences and the last assertion are proved. Assume now that (i) holds, and put $z = \bigvee^{A''_{[0,1]}}\{x \in A''_{[0,1]} : x \text{ satisfies } (5.2)\}$. Since $\{x \in A''_{[0,1]} : x \text{ satisfies } (5.2)\}$ is bounded and nonempty, we have $z = \bigvee^{A''_{\overline{\mathbb{R}}}}\{x \in A'' : x \text{ satisfies } (5.2)\}$ by Corollary 2.4 (a). Since ψ is a surjective isomorphism of complete lattices, it is sup-preserving and so $\psi(z)$ satisfies (5.2); hence $\psi(\overline{z})$ satisfies also (5.2) by Theorem 3.3 (i), which implies $\overline{z} = z$, and (i) holds with governing operator z. If (ii) holds, then by Theorem 2.3 and Theorem 3.3 (iii), (i) holds with governing operator $(\psi^{\mathcal{G}})^{-1}(\gamma)$. □

References

[1] C. A. Akemann. The general Stone-Weierstrass problem, *J. Funct. Anal. 4 (1969), 277-294.*

[2] C. A. Akemann, S. Eilers. Noncommutative end theory, *Pacific J. Math. 185 (1998), 47-88.*

[3] C. A. Akemann, N. Weaver. Minimal upper bounds of commuting self-adjoint operators, *Proc. Amer. Math. Soc. 124 (1996), No. 11, 3469-3476.*

[4] W. Arveson. On groups of automorphisms of operators algebras, *J. Functional Analysis 15 (1974), 217-243.*

[5] G. Birkhoff. *Lattice Theory*, Amer. Math. Soc. Colloq. Publ., Vol. 25, Amer. Math. Soc. Providence RI 1967.

[6] L.G Brown. Semicontinuity and multipliers of C*-algebras, *Canad. J. Math. 40 (1988), No. 4, 865-988.*

[7] H. Comman. Capacities on C^*-algebras, *Infin. Dimens. Anal. Quant. Prob. Relat. Top. 6 (2003), No. 3, 373-388.*

[8] H. Comman. Criteria for large deviations, *Trans. Amer. Math. Soc. 355 (2003), No. 7, 2905-2923.*

[9] C. Dellacherie, P. A. Meyer. *Probabilités et Potentiel*, Chapter IX-XI, Hermann, 1983.

[10] A. Dembo, O. Zeitouni. *Large Deviations Techniques and Applications*, Second Edition, Springer, New-York, 1998.

[11] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, D. S. Scott. *A Compendium of Continuous Lattices*, Springer, Berlin-Heidelberg-New York 1980.

[12] U. Haagerup. Operators valued weights in von Neumann Algebras, *J. Funct. Anal. 32 (1979), 175-206.*

[13] T. Kato. Spectral order and a matrix limit theorem, *Linear and Multilinear Algebra 8 (1979), 15-19.*

[14] H. Kim. Middle semicontinuity for unbounded operators, *Rocky Mountain J. Math. 30 (2000), No. 1, 257-273.*

[15] M. P. Olson. The self-adjoint operators of a von Neumann algebra form a conditionally complete lattice, *Proc. Amer. Math. Soc., Vol. 28 (1971), No. 2, 537-544.*

[16] W. Vervaat. Random upper semicontinuous functions and extremal processes, in *Probability and Lattices*, CWI Tract 110 (1991), ed. W. Vervaat and H. Holwerda.

Department of Mathematics, University of Santiago de Chile, Bernardo O'Higgins 3363 Santiago, Chile.

E-mail address: hcomman@usach.cl