A view on Noncommutative Large Deviations from a theory of Noncommutative Capacities*

Henri Comman†

Abstract

We define a large deviation principle for a net of states on a C*-algebra as the convergence of a net of noncommutative capacities toward a bounded maxitive one. This extends the classical definition for a net of regular probability measures on a locally compact Hausdorff space.

1 Introduction and background material

A set-capacity (in the sense of [6]) on a locally compact Hausdorff space X is a map from the powerset of X to $[0, \infty]$ which is inner regular on all sets, outer regular on compact sets, and vanishing on $\emptyset$. The narrow topology on the set of set-capacities is the coarsest topology for which the evaluations $\gamma \mapsto \gamma(Y)$ are upper semicontinuous (usc) for all closed $Y \subset X$ and lower semicontinuous (lsc) for all open $Y \subset X$.

Recall that a net of probability measures on X satisfies a large deviation principle when the probabilities of some Borel sets converges exponentially fast to 0. The rate of convergence is given by a positive bounded usc function f (equivalently by the so-called rate function $-\log f$). A set-capacity γ is maxitive if for all family $\{G_i; i \in I\}$ of open sets, $\gamma(\bigcup_{i\in I} G_i) = \sup_{i\in I} \gamma(G_i)$. Identifying bounded maxitive set-capacities with positive bounded usc functions is the trick which allows to use the set-capacities space to formulate a large deviation principle as the convergence of a net of some set-capacities to a bounded maxitive one, this limit giving the rate function ([7]).

In [4], we develop a noncommutative theory of capacities which extends to the general context of C*-algebras the essential results of the topological theory of set-capacities as given in [6], [7], [8]. Here, we give an application of this theory, showing how it is possible to extend the preceding scheme about large deviations for a net of states on a C*-algebras.

*Partially supported by the Presidential Chair on Qualitative Analysis of Quantum Dynamical Systems.

†Department of Mathematics, Catholic University of Chile, Casilla 306, Santiago 22, Chile, E-mail: hcomman@mat.puc.cl

We recall now some basic notions of noncommutative topology. Let $\mathcal{U}$ be a C*-algebra, $\mathcal{U}^{**}$ its enveloping Von Neumann algebra, $\tilde{\mathcal{U}}$ the algebra obtained by adjoining a unit 1 to $\mathcal{U}$, and $\mathcal{U}_+$ the positive part of $\mathcal{U}$. We denote by $\mathcal{C}_0(X)$ the C*-algebra of complex continuous functions vanishing at infinity on a locally compact Hausdorff space X, and write $\mathcal{B}(\mathcal{H})$ for the set of bounded operators on a complex Hilbert space $\mathcal{H}$.

A projection $p \in \mathcal{U}^{**}$ is *open* if there is an increasing net in $\mathcal{U}_+$ converging strongly to p. A projection $p \in \mathcal{U}^{**}$ is *closed* if $1 - p$ is open, and a closed projection $p \in \mathcal{U}^{**}$ is *compact* if there exists $x \in \mathcal{U}_+$ such that $p \leq x$ ([3], [9]). It is well known that the infimum (resp. supremum) of a family of closed (resp. open) projections is a closed (resp. open) projection. For a self-adjoint element $x \in \mathcal{U}^{**}$, E^x_Y denotes the spectral projection of x corresponding to the borel set $Y \subset \mathbf{R}$. We say that x is *q-upper semicontinuous* (q-usc) if $E^x_{]-\infty,t[}$ is open for all $t \in \mathbf{R}$, and that x is *q-lower semicontinuous* (q-lsc) if $-x$ is q-usc ([3]).

Example 1 Let $\mathcal{U}$ be commutative, isomorphic to $\mathcal{C}_0(X)$ for some locally compact Hausdorff space X, and $\mathcal{M}_1$ the set of regular probability measures on X. For all $\mu \in \mathcal{M}_1$, define $M_\mu : L^\infty(X, \mu) \to \mathcal{B}(L^2(X, \mu))$ by $(M_\mu(f)g)(t) = f(t)g(t)$. Then, for all $\mu \in \mathcal{M}_1$, $M_{\mu|\mathcal{C}_0(X)}$ is a cyclic representation of $\mathcal{C}_0(X)$. By definition, $\mathcal{C}_0(X)^{**}$ is the strong closure of $M(\mathcal{C}_0(X))$ in $\mathcal{B}(\oplus_{\mu\in\mathcal{M}_1} L^2(X, \mu))$ where $M = \oplus_{\mu\in\mathcal{M}_1} M_{\mu|\mathcal{C}_0(X)}$. For all positive bounded usc (resp. lsc) function f on X, define $M(f) = \oplus_{\mu\in\mathcal{M}_1} M_\mu(f)$. Then, the operator $M(f)$ is positive and q-usc (resp. q-lsc), and each positive q-usc (resp. q-lsc) operator is obtained in this way. A projection $p \in \mathcal{U}^{**}$ is compact (resp. open, closed) if and only if there exists some compact (resp. open, closed) $Y \subset X$ such that $p = M(1_Y)$.

In the rest of the paper, $\mathcal{U}$ denotes a C*-algebra, and A (resp. B, C) the set of compact (resp. open, closed) projections in $\mathcal{U}^{**}$.

2 Capacities space

Definition 1 A map $\gamma : B \cup C \to [0, \infty]$ is a *capacity* if

(i) $\gamma(0) = 0$,

(ii) $\gamma(b) = \sup_{a\leq b, a\in A} \gamma(a)$ for all $b \in B$,

(iii) $\gamma(c) = \inf_{b\geq c, b\in B} \gamma(b)$ for all $c \in C$.

Let Γ denote the set of capacities. We say that $\gamma \in \Gamma$ is *maxitive* if $\gamma(\bigvee_{i\in I} b_i) = \bigvee_{i\in I} \gamma(b_i)$ for all $\{b_i; i \in I\} \subset B$.

Example 2 For all state ω on $\mathcal{U}$ and $t > 0$, ω^t defined by $\omega^t(p) = \omega(p)^t$ for all $p \in B \cup C$, is a capacity. It is proved in [1] that ω is inner regular on B (and so outer regular on C) when $1 \in \mathcal{U}$. Suppose $1 \notin \mathcal{U}$, and let p be a projection in $\mathcal{U}^{**}$. It is known that p is open if and only if p is open in $\tilde{\mathcal{U}}^{**} (\simeq \mathcal{U}^{**} \oplus \mathbf{C})$, and that p is compact if and only if p is closed in $\tilde{\mathcal{U}}^{**}$ ([3]). Then, working in $\tilde{\mathcal{U}}^{**}$ with the canonical extension $\tilde{\omega}$ makes the job.

Definition 2 The *narrow* topology on Γ is the coarsest topology for which the evaluations $\gamma \mapsto \gamma(p)$ are usc for all $p \in C$ and lsc for all $p \in B$.

Remark 1 A net (γ_α) in Γ converges narrowly to $\gamma \in \Gamma$ if and only if $\limsup\ \gamma_\alpha(c) \leq \gamma(c)$ for all $c \in C$, and $\liminf\ \gamma_\alpha(b) \geq \gamma(b)$ for all $b \in B$.

The next Theorem 1 gives the general form of a bounded maxitive capacity in terms of a q-usc positive operator. It is the key result which will allows us to formulate a large deviation principle for a net of states in terms of convergence in Γ. For the proof, we will need the following results:

- The Urysohn's lemma ([2], Lemma 3.1) states that if $a \in A$, $b \in B$ and $a \leq b$, then there exists $x \in \mathcal{U}_+$ such that $a \leq x \leq b$. It follows that $b' = E^x_{]2/3,1]} \in B$, $a' = E^x_{[1/3,1]} \in A$ with $a \leq b' \leq a' \leq b$, which gives the non commutative analogue of local compactness.
- For all $b \in B$, $b = \bigvee_{a \leq b, a \in A} a$. Indeed, let $x \in \mathcal{U}_+$ with $x \leq b$. Then, $x \leq E^x_{]0,1]} \leq b$ and $E^x_{]0,1]} = \bigvee_{\lambda > 0} E^x_{[\lambda,1]}$ with $E^x_{[\lambda,1]} \in A$. Since $b \in B$, there is an increasing net (x_α) in $\mathcal{U}_+$ converging strongly to b, and the result follows.

Theorem 1 *Let γ be a map from $B \cup C$ to $[0, \infty]$. Then the following properties are equivalent:*

(i) γ is a bounded maxitive capacity.

(ii) There exists a q-usc positive operator z such that

$$\forall p \in B \cup C, \qquad \gamma(p) = \sup\{\lambda \in \sigma(z); \forall \varepsilon > 0, pE^z_{]\lambda-\varepsilon,\lambda+\varepsilon]} \neq 0\} \quad (1)$$

We say that z represents γ. Furthermore, there is a unique q-usc positive operator which represents γ.

Proof. (i)$\Rightarrow$(ii) To each real λ we assign the projection $E'_\lambda = \bigvee\{b \in B; \gamma(b) \leq \lambda\}$ (with the convention $\sup \emptyset = 0$) . If $\lambda < \mu$, then $E'_\lambda \leq E'_\mu$. For all $\lambda < 0$ $E'_\lambda = 0$, and $E'_{\gamma(1)} = 1$. So, we have defined a bounded spectral family $\{E'_\lambda\}$, and the family $\{E_\lambda\}$ where $E_\lambda = \bigwedge_{\mu > \lambda} E'_\mu$ is a bounded right continuous spectral family. By spectral theorem, there exists a unique positive operator z on the universal Hilbert space whose (right continuous) spectral family is $\{E_\lambda\}$. Since $E_\lambda \in \mathcal{U}^{**}$ for all $\lambda \in \mathbf{R}$, $z \in \mathcal{U}^{**}_+$. Moreover z is q-usc; indeed, for all $0 < \lambda$, for all $0 < \varepsilon < \lambda$, choose μ such that $0 < \lambda - \varepsilon < \mu < \lambda$. Then, $E_{\lambda-\varepsilon} \leq E'_\mu \leq E_{[0,\lambda[}$ and since $\bigvee_{\varepsilon < \lambda} E_{\lambda-\varepsilon} = E_{[0,\lambda[}$, we obtain $E_{[0,\lambda[} = \bigvee_{\mu<\lambda} E'_\mu$ which is an open projection.

We will prove that

$$\forall p \in B \cup C, \quad \gamma(p) = \sup\{\lambda \in \sigma(z); \forall \varepsilon > 0, p(E_{\lambda+\varepsilon} - E_{\lambda-\varepsilon}) \neq 0\}.$$

First we prove the inequality " $\geq$ ". Let $p \in B \cup C$, and $\lambda \in \sigma(z)$ such that $p(E_{\lambda+\varepsilon} - E_{\lambda-\varepsilon}) \neq 0$ for all $\varepsilon > 0$. Then $p \not\leq E_{\lambda-\varepsilon}$, and therefore $p \not\leq E'_{\lambda-\varepsilon}$ which implies $\gamma(p) \geq \lambda - \varepsilon$ for all $\varepsilon > 0$ (this is clear if $p \in B$, and if $p \in C$ notice that $\bigvee\{c \in C; \gamma(c) < \lambda\} \leq E'_\lambda$ by Definition 1 (iii)). Thus, $\gamma(p) \geq \lambda$.

We prove now the inequality " $\leq$ ". First we show that $\gamma(p) \in \sigma(z)$ for all $p \in B \cup C$. By inner regularity on C and since $\sigma(z)$ is closed, it suffices to show that $\gamma(b) \in \sigma(z)$ for all $b \in B$. If there exists $b \in B$ such that $\gamma(b) \notin \sigma(z)$, then there exists $\varepsilon > 0$ such that $E_{\gamma(b)-\varepsilon} = E_{\gamma(b)+\varepsilon}$. Let μ_1, μ_2 such that $\gamma(b) - \varepsilon < \mu_1 < \gamma(b) < \mu_2 < \gamma(b) + \varepsilon$. Then, $E_{\gamma(b)-\varepsilon} \leq E'_{\mu_1} \leq E'_{\mu_2} \leq E_{\gamma(b)+\varepsilon}$ which implies $E'_{\mu_1} = E'_{\mu_2}$. But $b \leq E'_{\mu_2}$, and by maxitivity $\gamma(E'_{\mu_1}) \leq \mu_1$ implies $b \not\leq E'_{\mu_1}$, which gives the contradiction. Then, for all $p \in B \cup C$, $\gamma(p) \in \sigma(z)$. Suppose

$$\gamma(p) > \sup\{\lambda \in \sigma(z); \forall \varepsilon > 0, p(E_{\lambda+\varepsilon} - E_{\lambda-\varepsilon}) \neq 0\}$$

for some $p \in B \cup C$. Then, since $\gamma(p) \in \sigma(z)$, $p(E_{\gamma(p)+\varepsilon'} - E_{\gamma(p)-\varepsilon'}) = 0$ for some $\varepsilon' > 0$, with $E_{\gamma(p)+\varepsilon'} - E_{\gamma(p)-\varepsilon'} \neq 0$. Since $p \leq E'_{\gamma(p)+\varepsilon'} \leq E_{\gamma(p)+\varepsilon'}$, we obtain $p \leq E_{\gamma(p)-\varepsilon'}$. Now, choose μ with $\gamma(p) - \varepsilon' < \mu < \gamma(p)$ and obtain by maxitivity $\gamma(p) \leq \gamma(E'_\mu) \leq \mu < \gamma(p)$ which gives the contradiction.

(ii)$\Rightarrow$(i) We establish first two facts.

Fact 1: for all $p \in B \cup C$, $p \leq E_{\gamma(p)}$. Define $\lambda_0(p) = \inf\{\lambda \in \mathbf{R}; p \leq E_\lambda\}$. Notice that $\lambda_0(p) \in \sigma(z)$ (otherwise there exists $\varepsilon > 0$ such that $E_{\lambda_0(p)-\varepsilon} = E_{\lambda_0(p)+\varepsilon}$, and $p \leq E_{\lambda_0(p)-\varepsilon}$ gives the contradiction). Since the family $\{E_\lambda\}$ is right continuous, we obtain $p \leq E_{\lambda_0(p)}$. Suppose $\lambda_0(p) > \gamma(p)$. Then, $p(E_{\lambda_0(p)+\varepsilon} - E_{\lambda_0(p)-\varepsilon}) = 0$ for some $\varepsilon > 0$ with $E_{\lambda_0(p)+\varepsilon} - E_{\lambda_0(p)-\varepsilon} \neq 0$. Since $p \leq E_{\lambda_0(p)+\varepsilon}$, we have $p \leq E_{\lambda_0(p)-\varepsilon}$ which gives the contradiction. Thus, $p \leq E_{\lambda_0(p)} \leq E_{\gamma(p)}$.

Fact 2: for all $\lambda \in \mathbf{R}$, $\gamma(E_{[0,\lambda[}) \leq \lambda$ (recall that z is q-usc and so $\gamma(E_{[0,\lambda[})$ is well defined). Suppose $\gamma(E_{[0,\lambda[}) > \lambda$, then there exists $\mu \in \sigma(z)$ with $\gamma(E_{[0,\lambda[}) \geq \mu > \lambda$ and $E_{[0,\lambda[}(E_{\mu+\varepsilon} - E_{\mu-\varepsilon}) \neq 0$ for all $\varepsilon > 0$, which implies $E_{[0,\lambda[} \not\leq E_{\mu-\varepsilon}$. But $E_{[0,\lambda[} \leq E_\lambda \leq E_{\mu-\varepsilon}$ for ε sufficiently small, which gives the contradiction.

Now, we can show that γ is a map completely maxitive on B. Let $\{b_i; i \in I\} \subset B$. By Fact 1, $b_i \leq E_{\gamma(b_i)}$ for all $i \in I$, and so $\bigvee_{i \in I} b_i \leq E_{\bigvee_{i \in I} \gamma(b_i)}$. Thus, $\bigvee_{i \in I} b_i \leq E_{[0,\bigvee_i \lambda_0(b_i)+\varepsilon[}$ for all $\varepsilon > 0$. By Fact 2 and since γ is clearly increasing, we have $\gamma(\bigvee_{i \in I} b_i) \leq \gamma(E_{[0,\bigvee_{i \in I} \gamma(b_i)+\varepsilon[}) \leq \bigvee_{i \in I} \gamma(b_i) + \varepsilon$ for all $\varepsilon > 0$, and thus $\gamma(\bigvee_{i \in I} b_i) \leq \bigvee_{i \in I} \gamma(b_i)$.

Now, we show that γ is a capacity. Clearly, $\gamma(0) = 0$ (with the convention $\sup \emptyset = 0$). Let $b \in B$. By Urysohn's lemma, for all $a \in A$ with $a \leq b$ there exists $a' \in A$, $b_a \in B$, such that $a \leq b_a \leq a' \leq b$, and since $b = \bigvee_{a \leq b, a \in A} a$, we obtain $b = \bigvee_{a \leq b, a \in A} b_a$. By maxitivity, $\gamma(b) = \sup_{b_a, a \in A} \gamma(b_a) = \sup_{a \leq b, a \in A} \gamma(a)$, giving inner regularity on B.

Let $c \in C$ and $\varepsilon > 0$. By Fact 1, $c \leq E_{[0,\gamma(c)+\varepsilon[}$, and by Fact 2, $\gamma(c) \leq \gamma(E_{[0,\gamma(c)+\varepsilon[}) \leq \gamma(c) + \varepsilon$. Since $E_{[0,\gamma(c)+\varepsilon[} \in B$, it follows that $\gamma(c) = \inf_{c \leq b, b \in B} \gamma(b)$ and so γ is a capacity. We have shown that γ is a maxitive capacity which is bounded since $\gamma(1) = \|z\|$.

Now, suppose that two q-usc positive operators z_1, z_2 represent γ, with respective right continuous spectral families $\{E^1_\lambda\}$ and $\{E^2_\lambda\}$. By Fact 1, for all $\lambda \in \mathbf{R}$, for all $\varepsilon > 0$, $E^1_\lambda \leq E^1_{[0,\lambda+\varepsilon[} \leq E^2_{\gamma(E^1_{[0,\lambda+\varepsilon[})}$ and by Fact 2, $E^2_{\gamma(E^1_{[0,\lambda+\varepsilon[})} \leq E^2_{\lambda+\varepsilon}$ and so $E^1_\lambda \leq E^2_{\lambda+\varepsilon}$ for all $\varepsilon > 0$. By right continuity of the family $\{E^2_\lambda\}$, we obtain $E^1_\lambda \leq E^2_\lambda$ and by symmetry $E^1_\lambda = E^2_\lambda$ for all $\lambda \in \mathbf{R}$. $\square$

Proposition 1 *Let $\mathcal{U}$ be commutative, isomorphic to $\mathcal{C}_0(X)$ where X is a locally compact Hausdorff space, and f a positive bounded usc function on X. Then, for all open or closed $Y \subset X$,*

$$\sup_{t\in Y} f(t) = \sup\{\lambda \in \sigma(M(f));\quad \forall\varepsilon > 0,\quad M(1_Y)(E_{\lambda+\varepsilon} - E_{\lambda+\varepsilon}) \neq 0\}$$

where $\{E_\lambda\}$ is the spectral family associated to $M(f)$ (see Example 1).

Proof. The spectral family $\{E_\lambda\}$ of $M(f)$ is given in the following way: for all $\mu \in \mathcal{M}_1$, for all $h \in L^2(X,\mu)$, $E_\lambda(h) = 1_{\{f<\lambda\}}h$. Let $Y \subset X$ be open or closed, and suppose

$$\lambda_0 = \sup_{t\in Y} f(t) > \sup\{\lambda \in \sigma(M(f));\quad \forall\varepsilon > 0,\quad M(1_Y)(E_{\lambda+\varepsilon} - E_{\lambda+\varepsilon}) \neq 0\}.$$

Since $f(t) \in \sigma(M(f))$ for all $t \in X$, we have $\lambda_0 \in \sigma(M(f))$, and so there exists $\varepsilon_0 > 0$ such that $M(1_Y)(E_{\lambda_0+\varepsilon_0} - E_{\lambda_0-\varepsilon_0}) = 0$. Since $M(1_Y) \leq E_{\lambda_0+\varepsilon}$ for all $\varepsilon > 0$, we have $M(1_Y) \leq E_{\lambda_0-\varepsilon_0}$. This implies $\mu(1_Y) \leq \mu(1_{Y\cap\{t;f(t)<\lambda_0-\varepsilon\}})$ for all $\mu \in \mathcal{M}_1$, giving a contradiction if we take $\mu = \delta_{t_0}$ the Dirac measure at $t_0 \in Y$, with $\lambda_0 - f(t_0) < \varepsilon$.

Suppose now

$$\sup_{t\in Y} f(t) < \sup\{\lambda \in \sigma(M(f));\quad \forall\varepsilon > 0,\quad M(1_Y)(E_{\lambda+\varepsilon} - E_{\lambda-\varepsilon}) \neq 0\}.$$

Then, there exists $\lambda' \in \sigma(M(f))$ such that $\sup_{t\in Y} f(t) < \lambda'$, and $M(1_Y) \not\leq E_{\lambda'-\varepsilon}$ for all $\varepsilon > 0$. Thus, for all $\varepsilon > 0$, there exists $h = \{h_\mu; \mu \in \mathcal{M}_1\} \in \oplus_{\mu\in\mathcal{M}_1} L^2(X,\mu)$ such that $\langle h, M(1_Y)h\rangle > \langle h, E_{\lambda'-\varepsilon}(h)\rangle$, and so there exists $\mu' \in \mathcal{M}_1$ such that

$$\langle h_{\mu'}, M(1_Y)h_{\mu'}\rangle_{L^2(X,\mu')} > \langle h_{\mu'}, E_{\lambda'-\varepsilon}h_{\mu'}\rangle_{L^2(X,\mu')}.$$

Equivalently,

$$\mu'(1_Y \mid h_{\mu'} \mid^2) > \mu'(1_{\{t;f(t)<\lambda'-\varepsilon\}} \mid h_{\mu'} \mid^2)$$

giving a contradiction if we take ε satisfying $\sup_{t\in Y} f(t) < \lambda' - \varepsilon$. $\square$

3 Large deviation principle for states

Definition 3 Let (t_α) be a net in $]0,\infty[$ converging to 0. A net (ω_α) of states on $\mathcal{U}$ satisfies a *large deviation principle* with powers (t_α) if there exists a positive q-usc operator z such that:

(i) $\limsup \omega_\alpha(c)^{t_\alpha} \leq \sup\{\lambda \in \sigma(z); \forall\varepsilon > 0, cE^z_{]\lambda-\varepsilon,\lambda+\varepsilon]} \neq 0\}$ for all $c \in C$,

(ii) $\liminf \omega_\alpha(b)^{t_\alpha} \geq \sup\{\lambda \in \sigma(z); \forall\varepsilon > 0, bE^z_{]\lambda-\varepsilon,\lambda+\varepsilon]} \neq 0\}$ for all $b \in B$.

We say that z *governs* the large deviation principle.

According to Example 2, Theorem 1 and Remark 1, an equivalent definition in terms of capacities is given in the following:

Proposition 2 *Let (t_α) be a net in $]0,\infty[$ converging to 0. A net (ω_α) of states on $\mathcal{U}$ satisfies a large deviation principle with powers (t_α) if and only if there exists a bounded maxitive capacity γ such that $(\omega_\alpha^{t_\alpha})$ converges narrowly to γ. The operator representing γ governs the large deviation principle.*

Let $\mathcal{U}$ be commutative, isomorphic to $\mathcal{C}_0(X)$ where X is a locally compact Hausdorff space, and (t_α) a net in $]0,\infty[$ converging to 0. Recall ([5], [7]) that a net (μ_α) of regular probability measures on X satisfies a large deviation principle with powers (t_α) if there exists a positive usc function f on X such that:

(i) $\limsup \mu_\alpha(F)^{t_\alpha} \leq \sup_{t\in Y} f(t)$ for all closed $F \subset X$,

(ii) $\liminf \mu_\alpha(G)^{t_\alpha} \geq \sup_{t\in G} f(t)$ for all open $G \subset X$.

In this case, f (or equivalently the so-called rate function $-\log f$) is said governing the large deviation principle.

By Proposition 1 and Riesz representation theorem, Definition 3 and Proposition 2 extend the classical notion of large deviation principle for a net of regular probability measures on X.

References

[1] Akemann C.A., 1969. The general Stone-Weierstrass problem, *J. Funct. Anal., 4, pp. 277-294.*

[2] Akemann C.A., 1971. A Gelfand representation theory for C*-algebras, *Pac. J. Math. 39, pp. 1-11.*

[3] Brown L.G., 1988. Semicontinuity and multipliers of C*-algebras, *Canad. J. Math., Vol. XL, No. 4, pp. 865-988.*

[4] Comman H. 2000. A non commutative topological theory of capacities and applications, PhD thesis, Pontifical Catholic University of Chile.

[5] Deuschel J.d. and Stroock D.W., 1989. Large Deviations, *Academic Press.*

[6] O'Brien G. L. and Vervaat W., 1991. Capacities, large deviations and loglog laws, *Stable Processes and Related Topics, Birkhauser, pp. 43-84.*

[7] O'Brien G. L., 1996. Sequence of capacities with connections to large deviation theory, *Journal of Theoritical Probability, 9, pp. 19-35.*

[8] O'Brien G.L. and Watson S., 1998. Relative compactness for capacities, measures, upper semicontinuous functions and closed sets. *Journal of Theoritical Probability, 11, pp. 577-588.*

[9] Pedersen G.K., 1979. C*-algebras and their automorphism groups, *Academic Press, London.*