Large deviations for quantum Markov semigroups on the 2×2-matrix algebra

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Abstract

Let $(\mathcal{T}_{*t})$ be a predual quantum Markov semigroup acting on the full 2×2-matrix algebra and having an absorbing pure state. We prove that for any initial state ω, the net of orthogonal measures representing the net of states $(\mathcal{T}_{*t}(\omega))$ satisfies a large deviation principle in the pure state space, with a rate function given in terms of the generator, and which does not depend on ω. This implies that $(\mathcal{T}_{*t}(\omega))$ is faithful for all t large enough. Examples arising in weak coupling limit are studied.

1 Introduction

The integral representation of states on a unital separable C^*-algebra establishes that each state is the barycentre of a measure concentrated on the set of pure states P ([14]). There are in general various such representing measures, a class of which is the so called orthogonal measures. In the case of the algebra of compact operators on some separable complex Hilbert space $\mathcal{H}$, when the state ω is given by the positive trace-one operator ρ, to each diagonal form $\rho = \sum_{i=1}^{\infty} a_i |e_i\rangle\langle e_i|$ is associated the orthogonal measure $\mu = \sum_{i=1}^{\infty} a_i \delta_{\omega_{|e_i\rangle\langle e_i|}}$, where $\omega_{|e_i\rangle\langle e_i|}$ is the pure state given by the projection $|e_i\rangle\langle e_i|$ (note that such a measure is uniquely determined by ρ if and only if all eigenvalues are simple).

In this paper we study large deviations for nets of orthogonal measures, and in particular when these nets are given by a quantum Markov semigroup acting on the full 2×2-matrix algebra M_2. More precisely, let $(\mathcal{T}_t)$ be such a semigroup having an absorbing state ω_∞ (i.e., in physical terminology, $(\mathcal{T}_t)$ converges to the equilibrium), and let $(\mathcal{T}_{*t})$ denotes its predual semigroup. For each initial state ω, we consider the net of states $(\mathcal{T}_{*t}(\omega))$. Our main result establishes that when ω_∞ is pure, the net of orthogonal measures representing $(\mathcal{T}_{*t}(\omega))$ satisfies a large deviation principle in P with powers $(1/t)$; the rate function takes the values $\{0, \eta - a, +\infty\}$ where a, η are parameters given by the generator of $(\mathcal{T}_{*t})$, and in particular it does not depend on ω (Theorem 4). This gives an exponential rate of "purification" of the state $\mathcal{T}_{*t}(\omega)$ in terms of the generator, in the large deviation sense (i.e. the rate with which the mass assigned to sets not containing the limit state vanishes). This rate is given by the eigenvalues of the operator $\mathcal{J}^*(|e_1\rangle\langle e_1|)$, where e_1 is the unit vector determining ω_∞, and $\mathcal{J}$ a completely positive operator on M_2 appearing in the generator. As a consequence, we obtain an exponential rate of convergence of the semigroup on projections (Corollary 1); this

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result can be interpreted as a noncommutative large deviation principle as defined by the author in previous works, and the so-called "rate operator" is exactly $\mathcal{J}^*(|e_1\rangle\langle e_1|)$ (see Remark 1).

The proof rests essentially on two operator-theoretic ingredients: the first one is the well-known form of the generator of quantum Markov semigroups acting on all bounded operators on $\mathcal{H}$, and having a pure stationary state; the second one is a general result that we prove for these semigroups, when they act on M_2 and admit an absorbing state. It establishes that $\mathcal{T}_{*t}(\omega)$ is faithful for all states $\omega \neq \omega_\infty$ and all t large enough (Theorem 3); this property will be, in turn, recovered as a consequence of large deviations. By a compactness argument combined with large deviations techniques, this result allows us to reduce the proof of the general initial state case to the one given by $\frac{1}{2}I$, where I is the identity.

Although we are mainly interested by orthogonal measures arising from dynamics as above, we begin in section 2 by considering a general family of such measures ($\mathcal{H}$ infinite dimensional), for which we give sufficient conditions to have large deviations (Proposition 1). This requires recent results in large deviation theory, and in particular a notion of exponential τ-smoothness, weaker than the usual exponential tightness ([6]). We then specialize to the case where $\mathcal{H}$ is N-dimensional and the net of states is converging (Proposition 2).

The problem of large deviations for orthogonal measures given by the evolution of quantum Markov semigroups is posed even in absence of convergence; this contrasts with the usual approach where it is the distance to some limit state which is measured. As a motivation to study in this way the asymptotic behavior we can mention some models of information dynamics, where algorithms are represented by the semigroups, the input by the initial state and the output by the limit state, which is typically pure ([4]); the complexity of the state under the evolution is represented by its support. Indeed, our large deviations describe the rate with which this support decreases.

In fact, the method used here for the two-dimensional case gives some indications about possible extensions to higher dimensions. Since the main tools for the proofs are the representation of the generator given by Theorem 1 and large deviations techniques, which both are valid in higher dimensions, we could reasonably expect that similar results hold at least in finite dimensions when there exists an absorbing pure state (the infinite dimensional case is more delicate because of the non-compactness of the pure state space). A crucial argument in the proof uses the fact that the operator y in (1) is diagonalizable, which is a particular feature of dimension two; this suggests that in dimension N, some extra conditions on the generator may have to be add. Note that Proposition 2 shows that a strict N-dimensional analogue of the large deviation principle proved here would imply the convergence of eigenvectors. It is likely that such a large deviation principle implies its noncommutative counterpart, namely an exponential rate of convergence on projections (see Remark 1); in other words, part (b) of Corollary 1 should admit a generalization. Since any noncommutative large deviation principle admits a unique rate operator ([5], Proposition 5.2), a natural question arises: Is this rate operator still given by $\mathcal{J}^*(|e_1\rangle\langle e_1|)$? Or coming back to the classical setting: Do the eigenvalues of $\mathcal{J}^*(|e_1\rangle\langle e_1|)$ still correspond to the finite values of the rate function? On the other hand, the only hypothesis in Theorem 1 being the existence of a stationary pure state, a similar study can be made in more general situations when there is no absorbing state. For instance, when there is a set S_∞ of pure states such that for each initial state ω, $\mathcal{T}_{*t}(\omega)$ converges to some element of S_∞; we should then obtain various rate functions indexed by the elements of S_∞. This kind of semigroups belongs to the class of the so-called "generic" semigroups, which arise in an extended version of the weak coupling limit ([1]). A class of such semigroups admitting an absorbing pure state (i.e. S_∞ contains only one element) is studied in Section 4.

1.1 Notations and background material

1.1.1 Quantum Markov semigroups

Let $\mathcal{H}$ be a complex separable Hilbert space, and let $K(\mathcal{H})$ be the set of compact operators acting on $\mathcal{H}$. Let P be the pure state space of $K(\mathcal{H})$ provided with the weak* topology, and note that P is completely regular Hausdorff, and compact when $\mathcal{H}$ is finite dimensional. For each $x \in K(\mathcal{H})$, we denote by $\hat{x}$ the map defined on P by $\hat{x}(\omega) = \omega(x)$, and note that $\hat{x}$ is continuous. The full $N \times N$-matrix algebra is denoted by M_N. By convention, for any self-adjoint $\rho \in M_N$, the expression $\rho = \sum_{i=1}^{N} a_i |e_i\rangle\langle e_i|$ has to be considered as a formal sum (i.e., a_i can be zero), which means that the set $\{e_i : 1 \leq i \leq N\}$ is an orthonormal basis diagonalizing ρ, where each e_i is an eigenvector corresponding to the eigenvalue a_i.

Let ω_ρ denotes the state given by the positive trace-one operator ρ. When ρ admits a diagonal form $\rho = \sum_{i=1}^{\infty} a_i |e_i\rangle\langle e_i|$, we will consider the measure $\mu = \sum_{i=1}^{\infty} a_i \delta_{\omega_{|e_i\rangle\langle e_i|}}$, where the sum has to be understood in the sense of the weak topology for Borel measures on P. It is easy to see that μ is an orthogonal measure representing ω_ρ, in the sense of the theory of integral representation of states ([14]). Clearly, when $\mathcal{H}$ has dimension 2, each state distinct of $I/2$ is represented by a unique orthogonal measure. We shall use the following lemma whose proof is straightforward.

Lemma 1 *For any net (ω_t) of states on M_N and any state ω on M_N, the following statements are equivalent.*

(i) $\lim \omega_t = \omega$;

(ii) *ω (resp. ω_t) is represented by an orthogonal measure μ (resp. μ_t) such that $\lim \mu_t(\hat{x}) = \mu(\hat{x})$ for all $x \in M_N$;*

(iii) *$\lim \mu_t(\hat{x}) = \mu(\hat{x})$ ($x \in M_N$) for all orthogonal measures μ_t and μ representing ω_t and ω, respectively.*

By a quantum Markov semigroup acting on M_2, we mean a w^*-continuous one-parameter semigroup $(\mathcal{T}_t)_{t\geq 0}$ of completely positive linear maps on M_2 preserving the identity I. We denote by $(\mathcal{T}_{*t})$ the predual semigroup, and by $(\widetilde{\mathcal{T}_{*t}})$ the associated semigroup acting on M_2, obtained by identifying ω_ρ and ρ. In other words, $(\widetilde{\mathcal{T}_{*t}})$ is a strongly continuous semigroup of completely positive contractions on M_2 defined by the relation $\omega_{\widetilde{\mathcal{T}_{*t}}(\rho)} = \mathcal{T}_{*t}(\omega_\rho)$ for all states ω_ρ and extended by linearity. A state ω is stationary if $\mathcal{T}_{*t}(\omega) = \omega$ for all $t \geq 0$, and a state ω_∞ is said to be absorbing if $\lim \mathcal{T}_{*t}(\omega) = \omega_\infty$ for all states ω; note that an absorbing state is stationary.

In the following theorem, we collect various results that will be used in the sequel. They appear in [8], and are valid more generally for uniformly continuous quantum Markov semigroups acting on the algebra of all bounded operators on $\mathcal{H}$ infinite dimensional.

Theorem 1 *Let $(\mathcal{T}_t)$ be a quantum Markov semigroup on M_2 having a stationary pure state given by the projection $|e\rangle\langle e|$. Then there exist y, z_1, z_2 in M_2 such that the following hold.*

(a) The generator $\tilde{\mathcal{L}}_*$ *of* $(\tilde{\mathcal{T}}_{*t})$ *has the form*

$$\forall \rho \in M_2, \qquad \tilde{\mathcal{L}}_*(\rho) = y\rho + \rho y^* + \mathcal{J}(\rho), \tag{1}$$

where $\mathcal{J}$ *is defined on* M_2 *by* $\mathcal{J}(\rho) = \sum_{i=1}^{2} z_i \rho z_i^*$*.*

(b) $ye = y^*e = z_1e = z_2e = 0$*.*

(c) y *is the generator of a one-parameter semigroup of contractions* $(C_t)_{t\geq 0}$ *on* $\mathcal{H}$*, and the semigroup* $(\mathcal{S}_t)_{t\geq 0}$ *on* M_2 *defined by* $\mathcal{S}_t(\rho) = C_t\rho C_t^*$*, satisfies for all* $t \geq 0$*,*

$$\forall \rho \geq 0, \qquad 0 \leq \mathcal{S}_t(\rho) \leq \tilde{\mathcal{T}}_{*t}(\rho) \tag{2}$$

and

$$\tilde{\mathcal{T}}_{*t} = \mathcal{S}_t + \int_0^t \tilde{\mathcal{T}}_{*t-s}\mathcal{J}\mathcal{S}_s ds. \tag{3}$$

1.1.2 Large deviations

We recall now some large deviations results for a net $(\mu_t)_{t\geq 0}$ of Borel probability measures on a completely regular Hausdorff topological space X. For each $[-\infty, +\infty[$-valued Borel measurable function h on X, we put $\mu_t^{1/t}(e^{th}) = (\int_X e^{th(x)}\mu_t(dx))^{1/t}$, and define $\Lambda(h) = \log \lim \mu_t^{1/t}(e^{th})$ when the limit exists. By definition, (μ_t) satisfies a large deviation principle with powers $(1/t)$ if there exists a $[0, +\infty]$-valued lower semi-continuous function J on X such that

$$\limsup \mu_t^{1/t}(F) \leq \sup_{x\in F} e^{-J(x)} \qquad \text{for all closed } F \subset X$$

and

$$\sup_{x\in G} e^{-J(x)} \leq \liminf \mu_t^{1/t}(G) \qquad \text{for all open } G \subset X;$$

J is called the rate function for $(\mu_t^{1/t})$, and for each Borel set $A \subset X$ such that $\sup_{x\in \mathrm{Int}(A)} e^{-J(x)} = \sup_{x\in \overline{A}} e^{-J(x)}$, the limit $\lim \mu_t^{1/t}(A)$ exists and satisfies

$$\lim \mu_t^{1/t}(A) = \sup_{x\in A} e^{-J(x)} \tag{4}$$

(such a set A is called a J-continuity set). The following notions have been introduced in [6].

Definition 1 The net (μ_t) is exponentially *τ-smooth* if for all open covers $\{G_i : i \in I\}$ of X and for all $\varepsilon > 0$, there exists a finite set $\{G_{i_1}, ..., G_{i_N}\} \subset \{G_i : i \in I\}$ such that

$$\limsup \mu_t^{1/t}(X \backslash \bigcup_{1\leq j\leq N} G_{i_j}) < \varepsilon.$$

Definition 2 A class $\mathcal{A}$ of $[-\infty, +\infty[$-valued continuous functions on X is said to be *approximating* if for each $x \in X$, each open set G containing x, each real $s > 0$, and each real $r > 0$, $\mathcal{A}$ contains some function h satisfying

$$e^{-r}1_{\{x\}} \leq e^h \leq 1_G \vee e^{-s}.$$

Under exponential τ-smoothness, the existence of $\Lambda(\cdot)$ on some approximating class $\mathcal{A}$ is sufficient to get large deviations, with a rate function which can be expressed in terms of $\mathcal{A}$. However, with slight extra conditions on $\mathcal{A}$, this expression is substantially simplified ([6], Corollary 2 and Corollary 4). This variant is stated in the following theorem, and will be used in the proof of Proposition 1.

Theorem 2 *Let $\mathcal{A}$ be an approximating class of bounded above functions such that for each $x \in X$, each open set G containing x, and each real $s > 0$, $\mathcal{A}$ contains some function h satisfying*

$$1_{\{x\}} \leq e^h \leq 1_G \vee e^{-s}.$$

If (μ_t) is exponentially τ-smooth and if $\Lambda(h)$ exists for all $h \in \mathcal{A}$, then (μ_t) satisfies a large deviation principle with powers $(1/t)$ and rate function

$$J(x) = \sup_{h\in\mathcal{A}, h(x)=0} \{-\Lambda(h)\} \qquad \textit{for all } x \in X.$$

2 Large deviations for orthogonal measures

In this section, we first give sufficient conditions for a net of orthogonal measures to satisfy a large deviation principle (Proposition 1). Next, we specialize to the case where $\mathcal{H}$ is finite dimensional and the net of states is converging (Proposition 2).

Proposition 1 *Let $(\omega_t)_{t\geq 0}$ be a net of states on $K(\mathcal{H})$, where each ω_t is represented by the orthogonal measure $\mu_t = \sum_{i=0}^{\infty} a_{i,t}\delta_{\omega_{|e_{i,t}\rangle\langle e_{i,t}|}}$, and assume that the following conditions hold:*

(i) $\lim_N \limsup(\sum_{i=N}^{\infty} a_{i,t})^{1/t} = 0$.

(ii) The net $(\omega_{|e_{i,t}\rangle\langle e_{i,t}|})_{t\geq 0}$ converges in P, for all $i \in \mathbb{N}$.

(iii) $\lim a_{i,t}^{1/t}$ *exists for all $i \in \mathbb{N}$.*

Then, (μ_t) satisfies a large deviation principle with powers $(1/t)$ and rate function

$$J(\omega) = \sup_{h\in\mathcal{A}, h(\omega)=0} \{-\Lambda(h)\} \qquad \text{for all } \omega \in P, \tag{5}$$

where $\mathcal{A} = \bigcup_{\omega\in P} \mathcal{A}_\omega$ with $\mathcal{A}_\omega$ the set of all finite infima of elements in $\{-|\hat{x} - \hat{x}(\omega)| : x \in K(\mathcal{H})\}$.

Proof. By (ii), for each $i \in \mathbb{N}$ there exists $e_i \in \mathcal{H}$ such that $\lim \omega_{|e_{i,t}\rangle\langle e_{i,t}|} = \omega_{|e_i\rangle\langle e_i|}$. Let $\mathcal{G}_0$ be an open cover of P, and let for each $i \in \mathbb{N}$ some $G_i \in \mathcal{G}_0$ containing $\omega_{|e_i\rangle\langle e_i|}$. By (i), for each $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $\limsup(\sum_{i=N_\varepsilon+1}^{\infty} a_{i,t})^{1/t} < \varepsilon$. Since for each $t \geq 0$,

$$\mu_t(P\backslash \cup_{i=0}^{N_\varepsilon} G_i) = \sum_{i=0}^{N_\varepsilon} a_{i,t}\delta_{\omega_{|e_{i,t}\rangle\langle e_{i,t}|}}(P\backslash \cup_{i=0}^{N_\varepsilon} G_i) + \sum_{i=N_\varepsilon+1}^{\infty} a_{i,t}\delta_{\omega_{|e_{i,t}\rangle\langle e_{i,t}|}}(P\backslash \cup_{i=0}^{N_\varepsilon} G_i),$$

with G_i containing $\omega_{|e_{i,t}\rangle\langle e_{i,t}|}$ for all $i \in \{0, ..., N_\varepsilon\}$ and all t large enough, we get

$$\limsup \mu_t^{1/t}(P\backslash \cup_{i=0}^{N_\varepsilon} G_i) < \varepsilon.$$

This shows that (μ_t) is exponentially τ-smooth since $\mathcal{G}_0$ is arbitrary. For each $\omega \in P$, each open set $G \subset P$ containing ω, and each $s > 0$, by definition of the w^*-topology, there exists a finite set $K_{\omega,G,s} \subset K(\mathcal{H})$ such that

$$1_{\{\omega\}} \leq e^{-\sup_{x\in K_{\omega,G,s}} |\hat{x}-\hat{x}(\omega)|} \leq 1_G \vee e^{-s},$$

hence $\mathcal{A}$ is an approximating class for P satisfying the hypothesis of Theorem 2. For any $\omega \in P$ and any finite subset $K \subset K(\mathcal{H})$, there is some $N \in \mathbb{N}$ such that

$$\limsup \mu_t^{1/t}(e^{-t\sup_{x\in K} |\hat{x}-\hat{x}(\omega)|}) = \limsup(\sum_{i=1}^{N} a_{i,t} e^{-t\sup_{x\in K} |\langle e_{i,t}, xe_{i,t}\rangle - \omega(x)|})^{1/t}$$

$$= \sup_{1\leq i\leq N} \lim a_{i,t}^{1/t} e^{-\sup_{x\in K} |\langle e_{i,t}, xe_{i,t}\rangle - \omega(x)|} = \lim \mu_t^{1/t}(e^{-t\sup_{x\in K} |\hat{x}-\hat{x}(\omega)|}), \qquad (6)$$

where the first equality follows from (i) and the second from (ii) and (iii). Since ω and K are arbitrary, $\lim \mu_t^{1/t}(e^{th})$ exists and so $\Lambda(h)$ exists for all $h \in \mathcal{A}$. By Theorem 2, (μ_t) satisfies a large deviation principle with powers $(1/t)$ and rate function (5). □

Lemma 2 *Let $(\rho_t)_{t\geq 0}$ be a net of hermitian matrices in M_N converging in norm to a hermitian matrix $\rho = \sum_{i=1}^{N} a_i |e_i\rangle\langle e_i|$ with $a_1 > ... > a_N$. Then for each t large enough ρ_t admits a diagonal form $\rho_t = \sum_{i=1}^{N} a_{i,t}|e_{i,t}\rangle\langle e_{i,t}|$ such that* $\lim a_{i,t} = a_i$ *and* $\lim |e_{i,t}\rangle\langle e_{i,t}| = |e_i\rangle\langle e_i|$ *(in norm) for all $i \in \{1, ..., N\}$.*

Proof. Let $m_{i,t}$ denote the multiplicity of $a_{i,t}$ $(1 \leq i \leq N)$. Since ρ_t converges in trace-norm, the assertion concerning the eigenvalues follows from the well-known inequality $\sum_{i=1}^{N} |a_i - a_{i,t}| \leq ||\rho - \rho_t||_1$ where $||\cdot||_1$ denotes the trace norm ([13]). Let $\varepsilon < \frac{1}{2}\min_{i=1}^{N-1}\{|a_i - a_{i+1}|\}$. Since (ρ_t) is uniformly converging, it converges in norm resolvent sense, so that $E^{\rho_t}_{]a_i-\varepsilon,a_i+\varepsilon[}$ converges uniformly to $E^{\rho}_{]a_i-\varepsilon,a_i+\varepsilon[} = E^{\rho}_{\{a_i\}}$ for each $i \in \{1, ..., N\}$. For each $\varepsilon' \leq \varepsilon/4$ we have

$$E^{\rho}_{\{a_1\}} = E^{\rho}_{]a_1-\varepsilon,a_1+\varepsilon[} = \lim_t E^{\rho_t}_{]a_1-\varepsilon,a_1+\varepsilon[} \sum_{j=1}^{N} \frac{1}{m_{j,t}} E^{\rho_t}_{]a_{j,t}-\varepsilon',a_{j,t}+\varepsilon'[}$$

$$= \lim_t \sum_{j=1}^{N} \frac{1}{m_{j,t}} E^{\rho_t}_{]a_1-\varepsilon,a_1+\varepsilon[\cap]a_{j,t}-\varepsilon',a_{j,t}+\varepsilon'[} = \lim_t \frac{1}{m_{1,t}} \sum_{j=1}^{m_{1,t}} E^{\rho_t}_{]a_1-\varepsilon,a_1+\varepsilon[\cap]a_{j,t}-\varepsilon',a_{j,t}+\varepsilon'[}$$

$$= \lim_t \frac{1}{m_{1,t}} \sum_{j=1}^{m_{1,t}} E^{\rho_t}_{]a_{j,t}-\varepsilon',a_{j,t}+\varepsilon'[} = \lim_t E^{\rho_t}_{]a_{1,t}-\varepsilon',a_{1,t}+\varepsilon'[} = \lim_t E^{\rho_t}_{\{a_{1,t}\}}.$$

Similarly we get $\lim_t E^{\rho_t}_{\{a_{i,t}\}} = E^{\rho}_{\{a_i\}}$ for all $i \in \{2, ..., N\}$, which proves the lemma. □

Part (a) of the following proposition shows that when (ω_{ρ_t}) converges to some state ω_{ρ_∞}, and under some extra condition on eigenvectors, large deviations for a suitable representing net of orthogonal measures are determined by the asymptotic behavior of the eigenvalues of ρ_t. The interesting case occurs when ω_{ρ_∞} is not faithful, otherwise the rate function (7) is trivial since $r = N$; it gives then the rate with which the support of ρ_t gets smaller. Note that by Lemma 2 the hypotheses are always satisfied in dimension 2 when $\omega_{\rho_\infty} \neq \frac{1}{2}I$, and in particular when ω_∞ is pure. Although this will not be used in the sequel, it is worth noticing that (assuming $\lim \omega_{\rho_t} = \rho_\infty$) a large deviation principle with rate function (7) implies the convergence of some eigenvectors, as establishes (b).

Proposition 2 *Let* $(\omega_{\rho_t})_{t\geq 0}$ *be a net of states on* M_N*, and assume that* (ω_{ρ_t}) w^**-converges to some state* ω_{ρ_∞}*. Let* $a_{1,t} \geq ... \geq a_{N,t}$ *be the eigenvalues of* ρ_t*, and* $a_1 \geq ... \geq a_r$ *be the non-zero eigenvalues of* ρ_∞ *counted with multiplicity (*$a_i = 0$ *for* $i > r$*), and consider the following property* (P_i) *for any* $i \in \{1, ..., N\}$*.*

(P_i) *For each* t *large enough* $a_{i,t}$ *(resp.* a_i*) admits an eigenvector* $e_{i,t}$ *(resp.* e_i*) such that* $\lim \omega_{|e_{i,t}\rangle\langle e_{i,t}|} = \omega_{|e_i\rangle\langle e_i|}$*.*

Then,

(a) If (P_i) *holds for all* $i \in \{1, ..., N\}$*, then the associated net* (μ_t) *of orthogonal measures satisfies a large deviation principle with powers* $(1/t)$ *if and only if* $\lim \frac{1}{t} \log a_{i,t}$ *exists for all* $i \in \{r+1, ..., N\}$*. In this case, the rate function is*

$$J(\omega_{|e\rangle\langle e|}) = \begin{cases} 0 & \text{if } |e\rangle\langle e| \in \{|e_i\rangle\langle e_i| : 1 \leq i \leq r\} \\ -\lim \frac{1}{t}\log a_{i,t} & \text{if } |e\rangle\langle e| = |e_i\rangle\langle e_i|,\ r+1 \leq i \leq N \\ +\infty & \textit{elsewhere.} \end{cases} \tag{7}$$

In particular, for each $i \in \{1, ..., N\}$ *and for each open set* $G \subset P$ *containing* $\omega_{|e_i\rangle\langle e_i|}$ *such that* $\overline{G} \cap \{\omega_{|e_j\rangle\langle e_j|} : 1 \leq j \leq N, j \neq i\} = \emptyset$*,* $\lim \frac{1}{t} \log \mu_t(G)$ *exists and satisfies*

$$\lim \frac{1}{t} \log \mu_t(G) = \lim \frac{1}{t} \log a_{i,t}.$$

(b) Conversely, if (ω_{ρ_t}) *is represented by a net of orthogonal measures* (μ_t) *satisfying a large deviation principle with rate function (7) (where* $\rho_\infty = \sum_{i=1}^N a_i |e_i\rangle\langle e_i|$*), then* (P_i) *holds for all* i *where* $J(\omega_{|e_i\rangle\langle e_i|}) < +\infty$*.*

Proof. Assume that (P_i) holds for all $i \in \{1, ..., N\}$. The convergence of states implies $\lim ||\rho_t - \rho_\infty|| = 0$ so that $\lim a_{i,t} = a_i$ for all $i \in \{1, ..., N\}$, and in particular $\lim a_{i,t}^{1/t} = 1$ when $1 \leq i \leq r$. Assume that $\lim a_{i,t}^{1/t}$ exists for all $i \in \{r+1, ..., N\}$. All the hypotheses of Proposition 1 hold, and the large deviations follow for (μ_t), with rate function given by (5). Let $\omega_{|e\rangle\langle e|} \in P$. For each $h \in \mathcal{A}$ with $h(\omega_{|e\rangle\langle e|}) = 0$ there exist $\omega' \in P$ and a finite set $K \subset M_N$ such that $h = \inf_{x \in K}\{-|\hat{x} - \hat{x}(\omega')|\}$ and $\omega_{|e\rangle\langle e|}(x) = \omega'(x)$ for all $x \in K$. We put $h_{K,\omega'} = h$, and $c_i = \lim \frac{1}{t}\log a_{i,t}$ for all $i \in \{1, ..., N\}$. By (6) we have

$$\Lambda(h_{K,\omega'}) = \sup_{1\leq i\leq N}(c_i - \sup_{x\in K} \lim |\langle e_{i,t}, x e_{i,t}\rangle - \omega'(x)|) = \sup_{1\leq i\leq N}(c_i - \sup_{x\in K} |\langle e_i, x e_i\rangle - \langle e, xe\rangle|), \tag{8}$$

so that $\Lambda(h) = 0$ for all $h \in \mathcal{A}$ when $|e\rangle\langle e| \in \{|e_i\rangle\langle e_i| : 1 \leq i \leq r\}$ (since in this case $c_i = 0$), hence $J(\omega_{|e\rangle\langle e|}) = 0$ by (5). Assume now that $|e\rangle\langle e| \notin \{|e_i\rangle\langle e_i| : 1 \leq i \leq N\}$. Let $x \in M_N$ such that $\langle e_i, xe_i\rangle \neq \langle e, xe\rangle$ for all $i \in \{1, ..., N\}$, and put $\delta = \inf_{1\leq i\leq N} |\langle e_i, xe_i\rangle - \langle e, xe\rangle|$. For all $M > 0$ there exists $r_M > 0$ such that $\inf |\langle e_i, r_M x e_i\rangle - \langle e, r_M x e\rangle| > M$, and since by (8)

$$-\Lambda(h_{\{r_M x\},\omega'}) \geq \inf_{1\leq i\leq N} |\langle e_i, r_M x e_i\rangle - \langle e, r_M x e\rangle|,$$

we get by letting $M \to +\infty$,

$$+\infty = \sup_M\{-\Lambda(h_{\{r_M x\},\omega'})\} \leq \sup_{h\in\mathcal{A}, h(\omega_{|e\rangle\langle e|})=0}\{-\Lambda(h)\},$$

that is $J(\omega_{|e\rangle\langle e|}) = +\infty$. Assume now that $|e\rangle\langle e| = |e_i\rangle\langle e_i|$ for some $i \in \{r+1, ..., N\}$. By the extended version of Varadhan' theorem for $[-\infty, +\infty[$-valued bounded above functions (see Corollary 3.2 of [7]), large deviations imply that $\lim \mu_t^{1/t}(\widehat{x}^t)$ exists for all positive $x \in M_N$, and satisfies $\lim \mu_t^{1/t}(\widehat{x}^t) = \sup_{\omega \in P} \widehat{x}(\omega) e^{-J(\omega)}$. Taking $x = |e_i\rangle\langle e_i|$ yields

$$\lim \mu_t^{1/t}(\widehat{|e_i\rangle\langle e_i|}^t) = \sup_{u \in \mathcal{H}, \|u\|=1} |\langle u, e_i\rangle|^2 e^{-J(\omega_{|u\rangle\langle u|})},$$

and by the preceding cases we see that the only possible non-zero value of the map $|u\rangle\langle u| \mapsto |\langle u, e_i\rangle|^2 e^{-J(\omega_{|u\rangle\langle u|})}$ is obtained at the point $|e_i\rangle\langle e_i|$, so that

$$\lim \mu_t^{1/t}(\widehat{|e_i\rangle\langle e_i|}^t) = e^{-J(\omega_{|e_i\rangle\langle e_i|})}.$$

Since $\lim\langle e_i, e_{j,t}\rangle = 0$ for all $j \neq i$ and $\lim\langle e_i, e_{i,t}\rangle = 1$, it follows that

$$\lim \mu_t^{1/t}(\widehat{|e_i\rangle\langle e_i|}^t) = \max_{1 \leq j \leq N} \{\lim a_{j,t}^{1/t} |\langle e_i, e_{j,t}\rangle|^2\} = \lim a_{i,t}^{1/t}$$

hence $J(\omega_{|e_i\rangle\langle e_i|}) = -\lim \frac{1}{t} \log a_{i,t}$. We have proved the "if" part of the first assertion of (a), and the second assertion. If (μ_t) satisfies a large deviation principle with powers $(1/t)$, then $\lim \mu_t^{1/t}(\widehat{|e_i\rangle\langle e_i|}^t)$ exists and

$$\lim \mu_t^{1/t}(\widehat{|e_i\rangle\langle e_i|}^t) = \max_{1 \leq j \leq N} \{\limsup a_{j,t}^{1/t} |\langle e_i, e_{j,t}\rangle|^2\} = \lim a_{i,t}^{1/t} \tag{9}$$

for all $i \in \{r+1, ..., N\}$; this proves the "only" part of the first assertion of (a). The last assertion follows from (4) since (7) implies that any open set $G \subset P$ containing $|e_i\rangle\langle e_i|$ with $\overline{G} \cap \{|e_j\rangle\langle e_j| : 1 \leq j \leq N, j \neq i\} = \emptyset$, is a J-continuity set. The proof of (a) is complete.

Assume that the hypotheses of (b) hold. The extended version of Varadhan's theorem together with (7) yield for each $i \in \{1, ..., N\}$,

$$\lim \mu_t^{1/t}(\widehat{|e_i\rangle\langle e_i|}^t) = \max_{1 \leq j \leq N} \{\limsup a_{j,t}^{1/t} |\langle e_i, e_{j,t}\rangle|^2\} \tag{10}$$

$$= \sup_{u \in \mathcal{H}, \|u\|=1} |\langle u, e_i\rangle|^2 e^{-J(\omega_{|u\rangle\langle u|})} = \lim a_{i,t}^{1/t}.$$

Let s be the greatest integer less than N such that $\lim a_{i,t}^{1/t} > 0$ for all $i \leq s$, and note that $s \geq r \geq 1$. By (10) we have $\lim a_{1,t}^{1/t} = \lim a_{j,t}^{1/t} |\langle e_1, e_{j,t}\rangle|^2$ for some $j \in \{1, ..., N\}$. If $j = 1$, then clearly $\lim |\langle e_1, e_{1,t}\rangle| = 1$ and the conclusion holds. If $s = 1$ the proof is complete so let us assume that $s > 1$. Assume that $j > 1$. Since $\lim a_{1,t}^{1/t} \geq \lim a_{j,t}^{1/t}$ we have necessarily $\lim a_{1,t}^{1/t} = \lim a_{j,t}^{1/t}$ and $\lim |\langle e_1, e_{j,t}\rangle| = 1$. With the new labeling of the eigenvalues of ρ_t obtained interchanging $(1, t)$ and (j, t), the conclusion holds for the first eigenvector. Suppose that the result holds for all $i - 1 < s$, and assume that

$$\lim a_{i,t}^{1/t} = \lim a_{j,t}^{1/t} |\langle e_i, e_{j,t}\rangle|^2$$

for some $j \in \{1, ..., N\}$. Since $\lim |e_{j,t}\rangle\langle e_{j,t}| = |e_j\rangle\langle e_j|$ for all $j \leq i - 1$ and $\lim a_{i,t}^{1/t} > 0$, it follows that $j \geq i$, and the proof is the same as for the case $i = 1$. We conclude with a finite recurrence (note that at each step the new labeling preserves the inequalities $\lim a_{i,t}^{1/t} \geq \lim a_{j,t}^{1/t}$ when $j \geq i$). □

Proposition 3 *Let $(\omega_t)_{t\geq 0}$ be a net of states on M_N. If (ω_t) is represented by a net of orthogonal measures which satisfies a large deviation principle with a rate function vanishing at a unique point $\omega_{|e\rangle\langle e|}$, then $\lim \omega_t = \omega_{|e\rangle\langle e|}$.*

Proof. Let $\mu_t = \sum_{1\leq i\leq N} a_{i,t}\delta_{\omega_{|e_{i,t}\rangle\langle e_{i,t}|}}$ be the orthogonal measure representing ω_t as above, let $\varepsilon > 0$, let $x \in M_N$ with $||x|| = 1$, and let G be the open neighborhood of $\omega_{|e\rangle\langle e|}$ defined by $\{\omega \in P : |\omega(x) - \omega_{|e\rangle\langle e|}(x)| < \varepsilon\}$. Then,

$$\mu_t(\hat{x}) = \sum_{1\leq i\leq N, \omega_{|e_{i,t}\rangle\langle e_{i,t}|}\in G} a_{i,t}\omega_{|e_{i,t}\rangle\langle e_{i,t}|}(x) + \sum_{1\leq i\leq N, \omega_{|e_{i,t}\rangle\langle e_{i,t}|}\notin G} a_{i,t}\omega_{|e_{i,t}\rangle\langle e_{i,t}|}(x) \tag{11}$$

and note that

$$\sum_{1\leq i\leq N, \omega_{|e_{i,t}\rangle\langle e_{i,t}|}\notin G} a_{i,t}\omega_{|e_{i,t}\rangle\langle e_{i,t}|}(x) \leq \mu_t(P\backslash G). \tag{12}$$

We have

$$\begin{aligned} \omega_{|e\rangle\langle e|}(x) \sum_{1\leq i\leq N, \omega_{|e_{i,t}\rangle\langle e_{i,t}|}\in G} a_{i,t} - \varepsilon \sum_{1\leq i\leq N, \omega_{|e_{i,t}\rangle\langle e_{i,t}|}\in G} a_{i,t} \leq \\ \sum_{1\leq i\leq N, \omega_{|e_{i,t}\rangle\langle e_{i,t}|}\in G} a_{i,t}\omega_{|e_{i,t}\rangle\langle e_{i,t}|}(x) \leq \sum_{1\leq i\leq N, \omega_{|e_{i,t}\rangle\langle e_{i,t}|}\in G} a_{i,t}(\omega_{|e\rangle\langle e|}(x) + \varepsilon) \\ \leq \omega_{|e\rangle\langle e|}(x) + \varepsilon. \end{aligned}$$

Let $\varepsilon \to 0$ and get

$$\omega_{|e\rangle\langle e|}(x) \sum_{1\leq i\leq N, \omega_{|e_{i,t}\rangle\langle e_{i,t}|}\in G} a_{i,t} \leq \sum_{1\leq i\leq N, \omega_{|e_{i,t}\rangle\langle e_{i,t}|}\in G} a_{i,t}\omega_{|e_{i,t}\rangle\langle e_{i,t}|}(x) \leq \omega_{|e\rangle\langle e|}(x) \tag{13}$$

The large deviations and the hypothesis on the rate function imply that $\lim \mu_t(P\backslash G) = 0$ (exponentially fast) hence

$$\lim_t \sum_{1\leq i\leq N, \omega_{|e_{i,t}\rangle\langle e_{i,t}|}\in G} a_{i,t} = \lim_t \mu_t(G) = 1,$$

and (13) yields

$$\lim_t \sum_{1\leq i\leq N, \omega_{|e_{i,t}\rangle\langle e_{i,t}|}\in G} a_{i,t}\omega_{|e_{i,t}\rangle\langle e_{i,t}|}(x) = \omega_{|e\rangle\langle e|}(x). \tag{14}$$

Then (11), (12), (14) give $\lim \mu_t(\hat{x}) = \omega_{|e\rangle\langle e|}(x)$, which proves the proposition by Lemma 1. $\square$

3 The case of states arising from quantum Markov semigroups on M_2

In this section, we first prove a general property of quantum Markov semigroups on M_2 having an absorbing state (Theorem 3). Looking the state space of M_2 as the unit ball in $\mathbb{R}^3$, it says that when the absorbing state ω_∞ is on the sphere (i.e., pure), $\mathcal{T}_{*t}(\omega)$ approaches ω_∞ remaining inside the open unit ball, for all states $\omega \neq \omega_\infty$. This property is crucial for the proof of the large deviations result (Theorem 4).

Theorem 3 *Let $(\mathcal{T}_t)$ be a quantum Markov semigroup acting on M_2, and having an absorbing state ω_∞. Then, for each state $\omega \neq \omega_\infty$, $\mathcal{T}_{*t}(\omega)$ is faithful for all t large enough.*

Proof. Clearly, the conclusion holds when ω_∞ is faithful, so that let us assume that ω_∞ is pure given by some projection $|e_1\rangle\langle e_1|$, and let e_2 be a unit vector orthogonal to e_1. First assume that ω is a pure state given by a unit vector $e = \alpha_1 e_1 + \alpha_2 e_2$ with $\alpha_2 \neq 0$, and suppose the conclusion does not hold. There exists a sequence (t_n) such that $\tilde{\mathcal{T}}_{*t_n}(|e\rangle\langle e|)$ is a rank-one projection for all $n \in \mathbb{N}$. Let $(\mathcal{S}_t), \mathcal{J}, y, z_1, z_2$ as in Theorem 1, and note that $ye_1 = y^* e_1 = 0$ implies that y is normal and diagonalizes in the basis $\{e_1, e_2\}$, hence $ye_2 = \gamma e_2$ for some $\gamma \in \mathbb{C}$. Put $-\eta = \gamma + \overline{\gamma}$, and note that $-\eta < 0$ since $-\eta$ is the least eigenvalue of $y + y^*$. For each $n \in \mathbb{N}$, let u_n be a unit vector such that $\langle u_n, \tilde{\mathcal{T}}_{*t_n}(|e\rangle\langle e|) u_n\rangle = 0$, and get by (2) and (3)

$$\sum_{i=1}^{2} \int_0^{t_n} \langle u_n, \tilde{\mathcal{T}}_{*t_n - s}(|z_i e^{sy} e\rangle\langle z_i e^{sy} e|) u_n\rangle ds = 0,$$

and so for $i \in \{1, 2\}$ and for each $s \in [0, t_n]$

$$\langle u_n, \tilde{\mathcal{T}}_{*t_n - s}(|z_i e^{sy} e\rangle\langle z_i e^{sy} e|) u_n\rangle = 0.$$

Since

$$|e^{sy} e\rangle\langle e^{sy} e| = |\alpha_1|^2 |e_1\rangle\langle e_1| + |\alpha_2|^2 e^{-s\eta} |e_2\rangle\langle e_2| + \alpha_1 \overline{\alpha_2} e^{s\overline{\gamma}} |e_1\rangle\langle e_2| + \alpha_2 \overline{\alpha_1} e^{s\gamma} |e_2\rangle\langle e_1|$$

we get for $i \in \{1, 2\}$

$$|z_i e^{sy} e\rangle\langle z_i e^{sy} e| = |\alpha_2|^2 e^{-s\eta} |z_i e_2\rangle\langle z_i e_2|$$

hence

$$\forall n \in \mathbb{N}, \forall s \in [0, t_n] \qquad \langle u_n, \tilde{\mathcal{T}}_{*s}(|z_i e_2\rangle\langle z_i e_2|) u_n\rangle = 0. \tag{15}$$

Put $z_i e_2 = \alpha_{1,i} e_1 + \alpha_{2,i} e_2$ for $i \in \{1, 2\}$ and get

$$\tilde{\mathcal{T}}_{*s}(|z_i e_2\rangle\langle z_i e_2|) = |\alpha_{1,i}|^2 |e_1\rangle\langle e_1| + |\alpha_{2,i}|^2 \tilde{\mathcal{T}}_{*s}(|e_2\rangle\langle e_2|) + \overline{\alpha_{2,i}} \alpha_{1,i} \tilde{\mathcal{T}}_{*s}(|e_1\rangle\langle e_2|)$$
$$+ \overline{\alpha_{1,i}} \alpha_{2,i} \tilde{\mathcal{T}}_{*s}(|e_2\rangle\langle e_1|). \tag{16}$$

By (3) we have

$$\tilde{\mathcal{T}}_{*s}(|e_2\rangle\langle e_2|) = e^{-s\eta} |e_2\rangle\langle e_2| + \sum_{i=1}^{2} \int_0^s \tilde{\mathcal{T}}_{*s-r}(z_i |e^{r\gamma} e_2\rangle\langle e^{r\gamma} e_2| z_i^*) dr$$

$$= e^{-s\eta} |e_2\rangle\langle e_2| + \sum_{i=1}^{2} \int_0^s e^{-r\eta} \tilde{\mathcal{T}}_{*s-r}(|z_i e_2\rangle\langle z_i e_2|) dr. \tag{17}$$

Then,

$$\tilde{\mathcal{T}}_{*s}(|e_1\rangle\langle e_2|) = \mathcal{S}_s(|e_1\rangle\langle e_2|) + \sum_{i=1}^{2} \int_0^s \tilde{\mathcal{T}}_{*s-r}(z_i |e^{ry} e_1\rangle\langle e^{ry} e_2| z_i^*) dr =$$

$$\mathcal{S}_s(|e_1\rangle\langle e_2|) + \sum_{i=1}^{2} \int_0^s \tilde{\mathcal{T}}_{*s-r}(|z_i e_1\rangle\langle z_i e^{r\gamma} e_2|) = \mathcal{S}_s(|e_1\rangle\langle e_2|) = e^{s\overline{\gamma}} |e_1\rangle\langle e_2|. \tag{18}$$

In the same way we get

$$\tilde{\mathcal{T}}_{*s}(|e_2\rangle\langle e_1|) = e^{s\gamma}|e_2\rangle\langle e_1|. \tag{19}$$

Then (16), (17), (18), (19) give for each $n \in \mathbb{N}$,

$$\begin{aligned}\langle u_n, \tilde{\mathcal{T}}_{*s}(|z_i e_2\rangle\langle z_i e_2|)u_n\rangle &= |\alpha_{1,i}|^2|\langle e_1, u_n\rangle|^2 + +2\mathcal{R}\mathrm{e}(\overline{\alpha_{1,i}}\alpha_{2,i}e^{s\gamma}\langle e_1, u_n\rangle\langle u_n, e_2\rangle) \\ &\quad +|\alpha_{2,i}|^2\langle u_n, \tilde{\mathcal{T}}_{*s}(|e_2\rangle\langle e_2|)u_n\rangle \\ &= |\alpha_{1,i}|^2|\langle e_1, u_n\rangle|^2 + 2\mathcal{R}\mathrm{e}(\overline{\alpha_{1,i}}\alpha_{2,i}e^{s\gamma}\langle e_1, u_n\rangle\langle u_n, e_2\rangle) + |\alpha_{2,i}|^2 e^{-s\eta}(|\langle e_2, u_n\rangle|^2 \\ &\quad +|\alpha_{2,i}|^2\sum_{j=1}^{2}\int_0^s e^{-r\eta}\langle u_n, \tilde{\mathcal{T}}_{*s-r}(|z_j e_2\rangle\langle z_j e_2|)u_n\rangle dr. \end{aligned} \tag{20}$$

By (15) and (20) we obtain for each $n \in \mathbb{N}$ and each $s \in [0, t_n]$

$$|\alpha_{1,i}|^2|\langle e_1, u_n\rangle|^2 + 2\mathcal{R}\mathrm{e}(\overline{\alpha_{1,i}}\alpha_{2,i}e^{s\gamma}\langle e_1, u_n\rangle\langle u_n, e_2\rangle) + |\alpha_{2,i}|^2 e^{-s\eta}|\langle e_2, u_n\rangle|^2 = 0. \tag{21}$$

Taking the limit when $n \to +\infty$ in (21) with $s = t_n$ yields

$$\lim_n |\alpha_{1,i}|^2|\langle e_1, u_n\rangle|^2 = 0, \tag{22}$$

and (15) with $s = 0$ gives

$$\forall n \in \mathbb{N}, \qquad \alpha_{1,i}\langle e_1, u_n\rangle + \alpha_{2,i}\langle e_2, u_n\rangle = 0. \tag{23}$$

First assume $\alpha_{2,i} = 0$. If $\alpha_{1,i} \neq 0$, then $|u_n\rangle\langle u_n| = |e_2\rangle\langle e_2|$ for all $n \in \mathbb{N}$ by (23), and (2) implies

$$\forall n \in \mathbb{N}, \qquad \langle e_2, S_{t_n}(|e\rangle\langle e|)e_2\rangle = \langle e^{t_n\overline{\gamma}}e_2, |e\rangle\langle e|e^{t_n\overline{\gamma}}e_2\rangle = |\alpha_2|^2 e^{-t_n\eta} = 0$$

which gives the contradiction since $\alpha_2 \neq 0$. It follows that $\alpha_{1,i} = 0$, and so $z_i = 0$. Assume now $\alpha_{2,i} \neq 0$. It is easy to see that (22) and (23) imply $\lim\langle e_2, u_n\rangle = 0$ and $\alpha_{1,i} = 0$, hence $|u_n\rangle\langle u_n| = |e_1\rangle\langle e_1|$ for all $n \in \mathbb{N}$, which gives the contradiction since $\lim\langle e_1, \tilde{\mathcal{T}}_{*t_n}(|e\rangle\langle e|)e_1\rangle = 1$. We obtain finally that (22) and (23) imply $z_1 = z_2 = 0$, that is $\tilde{\mathcal{T}}_{*t} = S_t$ for all $t \geq 0$. Since $\min\sigma(\tilde{\mathcal{T}}_{*t_n}(|e\rangle\langle e|)) = 0$ and $\mathrm{tr}(\tilde{\mathcal{T}}_{*t_n}(|e\rangle\langle e|)) = 1$, we have

$$\begin{aligned}1 = ||S_{t_n}(|e\rangle\langle e|)|| = \sup_{u\in\mathcal{H},||u||=1} \{&|\alpha_1|^2|\langle u, e_1\rangle|^2 + |\alpha_2|^2 e^{-t_n\eta}|\langle u, e_2\rangle|^2 \\ &+2\mathcal{R}\mathrm{e}(\overline{\alpha_1}\alpha_2 e^{t_n\gamma}\langle e_1, u\rangle\langle u, e_2\rangle)\},\end{aligned}$$

and letting $t_n \to +\infty$ it follows that

$$1 = \sup_{u\in\mathcal{H},||u||=1} \{|\alpha_1|^2|\langle u, e_1\rangle|^2\},$$

which implies $|\alpha_1| = 1$ and the contradiction since $\alpha_2 \neq 0$. The theorem is proved when ω is pure. Assume now that ω is given by some strictly positive operator ρ, i.e. $cI \leq \rho \leq I$ for some $c > 0$. Since $\mathcal{S}_t(cI) = ce^{t(y+y^*)}$, (2) gives

$$ce^{-t\eta} = \min\sigma(\mathcal{S}_t(cI)) \leq \min\sigma(\tilde{\mathcal{T}}_{*t}(\rho)),$$

which shows that $\mathcal{T}_{*t}(\omega)$ is faithful for all $t \geq 0$. $\square$

The following theorem is our large deviation result. It shows that when $(\mathcal{T}_t)$ has an absorbing pure state ω_∞, and for any initial state $\omega_\rho \neq \omega_\infty$, the net $(\mathcal{T}_{*t}(\omega_\rho))$ converges exponentially fast (in the large deviation sense expressed by (24)) toward ω_∞ with rate $\eta - a$, where η, a are parameters given by the generator. Note that (24) implies $a_{2,t,\rho} > 0$ for all t large enough, so that Theorem 4 contains Theorem 3.

Theorem 4 *Let $(\mathcal{T}_t)$ be a quantum Markov semigroup on M_2, and having an absorbing pure state $\omega_{|e_1\rangle\langle e_1|}$. Let y, z_1, z_2 be the parameters of the generator $\tilde{\mathcal{L}}_*$ as in Theorem 1, and let e_2 be a unit vector orthogonal to e_1. Then for each state $\omega_\rho \neq \omega_{|e_1\rangle\langle e_1|}$, the net $(\mu_{t,\rho})$ of orthogonal measures representing $(\mathcal{T}_{*t}(\omega_\rho))$ satisfies a large deviation principle with powers $(1/t)$ and rate function*

$$J(\omega_{|e\rangle\langle e|}) = \begin{cases} 0 & \text{if } |e\rangle\langle e| = |e_1\rangle\langle e_1| \\ \eta - a & \text{if } |e\rangle\langle e| = |e_2\rangle\langle e_2| \\ +\infty & \textit{elsewhere}, \end{cases}$$

where $a = |\langle z_1 e_2, e_2\rangle|^2 + |\langle z_2 e_2, e_2\rangle|^2$ and η is the greatest eigenvalue of $-(y + y^*)$; moreover, $\eta - a > 0$. In particular, the rate function does not depend on the initial state ω_ρ, and for each open sets $G \subset P$ containing $|e_2\rangle\langle e_2|$ such that $|e_1\rangle\langle e_1| \notin \overline{G}$, $\lim \frac{1}{t} \log \mu_{t,\rho}(G)$ exists and satisfies

$$\lim \frac{1}{t} \log \mu_{t,\rho}(G) = \lim \frac{1}{t} \log a_{2,t,\rho} = a - \eta, \tag{24}$$

where $a_{2,t,\rho}$ is the least eigenvalue of $\tilde{\mathcal{T}}_{*t}(\rho)$.

Proof. Let $\omega_\rho \neq \omega_{|e_1\rangle\langle e_1|}$ be a state, and let $\tilde{\mathcal{T}}_{*t}(\rho)$ have the diagonal form $\tilde{\mathcal{T}}_{*t}(\rho) = \sum_{i=1}^{2} a_{i,t,\rho} |e_{i,t,\rho}\rangle\langle e_{i,t,\rho}|$ as in Lemma 2. By Proposition 2 we only have to check that $\lim a_{2,t,\rho}^{1/t}$ exists and equals $e^{a-\eta}$. Since

$$\tilde{\mathcal{T}}_{*t}(I) = \tilde{\mathcal{T}}_{*t}(|e_1\rangle\langle e_1|) + \tilde{\mathcal{T}}_{*t}(|e_2\rangle\langle e_2|) = |e_1\rangle\langle e_1| + \tilde{\mathcal{T}}_{*t}(|e_2\rangle\langle e_2|), \tag{25}$$

by (17) we have

$$\begin{aligned} \min \sigma(\tilde{\mathcal{T}}_{*t}(I)) \leq \langle e_2, \tilde{\mathcal{T}}_{*t}(I) e_2\rangle &= \langle e_2, \tilde{\mathcal{T}}_{*t}(|e_2\rangle\langle e_2|) e_2\rangle \\ &= e^{-t\eta} + \sum_{i=1}^{2} \int_0^t e^{-s\eta} \langle e_2, \tilde{\mathcal{T}}_{*t-s}(|z_i e_2\rangle\langle z_i e_2|) e_2\rangle \end{aligned} \tag{26}$$

and by using the first equality of (20) (with e_2 in place of u_n and $t - s$ in place of s), the last above equality becomes

$$\langle e_2, \tilde{\mathcal{T}}_{*t}(|e_2\rangle\langle e_2|) e_2\rangle = e^{-t\eta} + a \int_0^t e^{-s\eta} \langle e_2, \tilde{\mathcal{T}}_{*t-s}(|e_2\rangle\langle e_2|) e_2\rangle$$

with $a = |\alpha_{2,1}|^2 + |\alpha_{2,2}|^2$. By putting $u(t) = \langle e_2, \tilde{\mathcal{T}}_{*t}(|e_2\rangle\langle e_2|) e_2\rangle$ for all $t \geq 0$, and applying the Laplace transform, it is easy to see that the equation

$$u(t) = e^{-t\eta} + a \int_0^t e^{-s\eta} u(t-s) ds$$

has the unique solution

$$\forall t \geq 0, \qquad u(t) = e^{t(a-\eta)}, \tag{27}$$

with $a - \eta < 0$ (since $\lim \langle e_2, \tilde{\mathcal{T}}_{*t}(|e_2\rangle\langle e_2|) e_2\rangle = 0$). It follows from (26) that

$$\limsup \min \sigma(\tilde{\mathcal{T}}_{*t}(I))^{1/t} \leq \lim \langle e_2, \tilde{\mathcal{T}}_{*t}(|e_2\rangle\langle e_2|) e_2\rangle^{1/t} = e^{a-\eta} < 1. \tag{28}$$

For each $t \geq 0$ let u_t be a unit vector such that

$$\min \sigma(\tilde{\mathcal{T}}_{*t}(I)) = \langle u_t, \tilde{\mathcal{T}}_{*t}(I) u_t\rangle.$$

By (25) we have

$$\min\sigma(\tilde{\mathcal{T}}_{*t}(I)) = |\langle u_t, e_1\rangle|^2 + \langle u_t, \tilde{\mathcal{T}}_{*t}(|e_2\rangle\langle e_2|)u_t\rangle.$$

Let (u_{t_j}) be a subnet of (u_t) such that

$$\liminf\min\sigma(\tilde{\mathcal{T}}_{*t}(I))^{1/t} = \lim\min\sigma(\tilde{\mathcal{T}}_{*t_j}(I))^{1/t_j}$$

and get

$$\liminf\min\sigma(\tilde{\mathcal{T}}_{*t}(I))^{1/t} = \limsup|\langle u_{t_j}, e_1\rangle|^{2/t_j} \vee \limsup\langle u_{t_j}, \tilde{\mathcal{T}}_{*t_j}(|e_2\rangle\langle e_2|)u_{t_j}\rangle^{1/t_j}. \tag{29}$$

Let (u_{t_k}) be a subnet of (u_{t_j}). Then $|u_{t_k}\rangle\langle u_{t_k}|$ has a subnet $|u_{t_l}\rangle\langle u_{t_l}|$ converging to some projection $|u\rangle\langle u|$. If $|u\rangle\langle u| \neq |e_2\rangle\langle e_2|$, then $\liminf\min\sigma(\tilde{\mathcal{T}}_{*t}(I))^{1/t} = 1$, which contradicts (28). Therefore, $|u\rangle\langle u| = |e_2\rangle\langle e_2|$ and since the subnet $|u_{t_k}\rangle\langle u_{t_k}|$ is arbitrary, $|u_t\rangle\langle u_t|$ converges to $|e_2\rangle\langle e_2|$. Put $u_t = b_{1,t}e_1 + b_{2,t}e_2$ for all $t \geq 0$, and get

$$\begin{aligned}\langle u_{t_j}, \tilde{\mathcal{T}}_{*t_j}(|e_2\rangle\langle e_2|)u_{t_j}\rangle = {} & |b_{1,t_j}|^2\langle e_1, \tilde{\mathcal{T}}_{*t_j}(|e_2\rangle\langle e_2|)e_1\rangle + |b_{2,t_j}|^2\langle e_2, \tilde{\mathcal{T}}_{*t_j}(|e_2\rangle\langle e_2|)e_2\rangle \\ & + \overline{b_{1,t_j}}b_{2,t_j}\langle e_1, \tilde{\mathcal{T}}_{*t_j}(|e_2\rangle\langle e_2|)e_2\rangle + \overline{b_{2,t_j}}b_{1,t_j}\langle e_2, \tilde{\mathcal{T}}_{*t_j}(|e_2\rangle\langle e_2|)e_1\rangle.\end{aligned} \tag{30}$$

Then (17) combined with (16), (18), (19) yield

$$\forall t \geq 0, \qquad \langle e_1, \tilde{\mathcal{T}}_{*t}(|e_2\rangle\langle e_2|)e_2\rangle = a\int_0^t e^{-s\eta}\langle e_1, \tilde{\mathcal{T}}_{*t-s}(|e_2\rangle\langle e_2|)e_2\rangle ds,$$

and by an application of Laplace transform we get as unique solution

$$\langle e_1, \tilde{\mathcal{T}}_{*t}(|e_2\rangle\langle e_2|)e_2\rangle = 0 \qquad \text{for all } t \geq 0. \tag{31}$$

Similar calculations yield

$$\langle e_2, \tilde{\mathcal{T}}_{*t}(|e_2\rangle\langle e_2|)e_1\rangle = 0 \qquad \text{for all } t \geq 0. \tag{32}$$

It follows from (27) and (30) that

$$\langle u_{t_j}, \tilde{\mathcal{T}}_{*t_j}(|e_2\rangle\langle e_2|)u_{t_j}\rangle \geq |b_{2,t_j}|^2 e^{t_j(a-\eta)}.$$

Since $\lim|u_{t_j}\rangle\langle u_{t_j}| = |e_2\rangle\langle e_2|$, we have $\lim|b_{2,t_j}| = 1$ and by (29)

$$\liminf\min\sigma(\tilde{\mathcal{T}}_{*t}(I))^{1/t} \geq \limsup\langle u_{t_j}, \tilde{\mathcal{T}}_{*t_j}(|e_2\rangle\langle e_2|)u_{t_j}\rangle^{1/t_j} \geq e^{a-\eta}. \tag{33}$$

Then (28) and (33) yield

$$\lim\min\sigma(\tilde{\mathcal{T}}_{*t}(I))^{1/t} = \lim\langle e_2, \tilde{\mathcal{T}}_{*t}(|e_2\rangle\langle e_2|)e_2\rangle^{1/t} = e^{(a-\eta)}. \tag{34}$$

If ρ is strictly positive, then $cI \leq \rho \leq I$ and

$$c\min(\sigma(\tilde{\mathcal{T}}_{*t}(I))) \leq \min(\sigma(\tilde{\mathcal{T}}_{*t}(\rho))) \leq \min(\sigma(\tilde{\mathcal{T}}_{*t}(I)))$$

for some $c > 0$, hence

$$\lim a_{2,t,\rho}^{1/t} = \lim\min(\sigma(\tilde{\mathcal{T}}_{*t}(\rho)))^{1/t} = \lim\min\sigma(\tilde{\mathcal{T}}_{*t}(I))^{1/t} = e^{a-\eta}. \tag{35}$$

Assume now that $\rho = |e\rangle\langle e|$ for some unit vector $e = \alpha_1e_1 + \alpha_2e_2$ with $\alpha_2 \neq 0$. Let $(a_{2,t_j,\rho}^{1/t_j})$ be a subsequence of $(a_{2,t,\rho}^{1/t})$, and consider the corresponding subsequence $(\mu_{t_j,\rho})$

of $(\mu_{t,\rho})$. Note that by Theorem 3, $a_{2,t_j,\rho} > 0$ for all j large enough. By a well-known compactness result in large deviation theory (see Lemma 4.1.23 of [10], or Corollary 5 of [6] for a general version), $(\mu_{t_j,\rho})$ has a subsequence $(\mu_{t_{j_k},\rho})$ satisfying a large deviation principle, so that $\lim a_{2,t_{j_k},\rho}^{1/t_{j_k}}$ exists by Proposition 2. Put $e^{-l} = \lim a_{2,t_{j_k},\rho}^{1/t_{j_k}}$, and get for each $k' \in \mathbb{N}$,

$$-l = \lim_k \frac{1}{t_{j_k} - t_{j_{k'}} + t_{j_{k'}}} \log \min \sigma(\tilde{\mathcal{T}}_{*t_{j_k} - t_{j_{k'}} + t_{j_{k'}}}(\rho)) =$$

$$\lim_k \frac{t_{j_k} - t_{j_{k'}}}{t_{j_k} - t_{j_{k'}} + t_{j_{k'}}} \cdot \frac{1}{t_{j_k} - t_{j_{k'}}} \log \min \sigma(\tilde{\mathcal{T}}_{*t_{j_k} - t_{j_{k'}}}(\tilde{\mathcal{T}}_{*t_{j_{k'}}}(\rho)))$$

$$\lim_k \frac{1}{t_{j_k} - t_{j_{k'}}} \log \min \sigma(\tilde{\mathcal{T}}_{*t_{j_k} - t_{j_{k'}}}(\tilde{\mathcal{T}}_{*t_{j_{k'}}}(\rho)).$$

Since $\lim_k (t_{j_k} - t_{j_{k'}}) = +\infty$, the sequence $(\tilde{\mathcal{T}}_{*t_{j_k} - t_{j_{k'}}}(\tilde{\mathcal{T}}_{*t_{j_{k'}}}(\rho)))_{k \in \mathbb{N}, t_{j_k} \geq t_{j'_k}}$ is a subsequence of $(\tilde{\mathcal{T}}_{*t}(\tilde{\mathcal{T}}_{*t_{j_{k'}}}(\rho))_{t>0}$. Since $\tilde{\mathcal{T}}_{*t_{j_{k'}}}(\rho)$ is strictly positive for k' large enough, (35) (with $\tilde{\mathcal{T}}_{*t_{j_{k'}}}(\rho)$ in place of ρ) implies $\lim a_{2,t,\tilde{\mathcal{T}}_{*t_{j_{k'}}}(\rho)}^{1/t} = e^{a-\eta}$ hence

$$-l = \lim \frac{1}{t_{j_k} - t_{j_{k'}}} \log \min \sigma(\tilde{\mathcal{T}}_{*t_{j_k} - t_{j_{k'}}}(\tilde{\mathcal{T}}_{*t_{j_{k'}}}(\rho)) =$$

$$\lim \frac{1}{t_{j_k} - t_{j_{k'}}} \log a_{2,t_{j_k} - t_{j_{k'}}, \tilde{\mathcal{T}}_{*t_{j_{k'}}}(\rho)} = a - \eta.$$

It follows that each subsequence of $(a_{2,t,\rho}^{1/t})$ has a subsequence converging to $e^{a-\eta}$, hence $\lim a_{2,t,\rho}^{1/t} = e^{a-\eta}$. □

The following result shows that the large deviation principle as well as the exponential rate of convergence on projections are given by the eigenvalues of $\mathcal{J}^*(|e_1\rangle\langle e_1|)$.

Corollary 1 *Let $(\mathcal{T}_t)$ be a quantum Markov semigroup on M_2 having an absorbing pure state $\omega_{|e_1\rangle\langle e_1|}$, let e_2 be a unit vector orthogonal to e_1, and let $\mathcal{J}$ be the operator on M_2 appearing in the generator (1). Then the following conclusions hold.*

(a) $\mathcal{J}^*(|e_1\rangle\langle e_1|) = (\eta - a)|e_2\rangle\langle e_2|$ with η, a as in Theorem 4;

(b) For each state $\omega \neq \omega_{|e_1\rangle\langle e_1|}$ and each projection $p \in M_2 \backslash \{0\}$ we have

$$\lim \frac{1}{t} \log \omega(\mathcal{T}_t(p)) = \begin{cases} a - \eta & \text{if } p = |e_2\rangle\langle e_2| \\ & \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Differentiating (27), (31), (32) and taking the value at $t = 0$ yields respectively $\langle e_2, \tilde{\mathcal{L}}_*(|e_2\rangle\langle e_2|)e_2\rangle = a - \eta$, $\langle e_1, \tilde{\mathcal{L}}_*(|e_2\rangle\langle e_2|)e_2\rangle = 0$, $\langle e_2, \tilde{\mathcal{L}}_*(|e_2\rangle\langle e_2|)e_1\rangle = 0$. By (27) and the preservation of the trace we have $\langle e_1, \mathcal{T}_{*t}(|e_2\rangle\langle e_2|)e_1\rangle = 1 - e^{t(a-\eta)}$ hence $\langle e_1, \tilde{\mathcal{L}}_*(|e_2\rangle\langle e_2|)e_1\rangle = \eta - a$. Then (a) follows by noting that

$$\mathcal{J}(I) = \mathcal{J}(|e_2\rangle\langle e_2|) = \tilde{\mathcal{L}}_*(|e_2\rangle\langle e_2|) + \eta|e_2\rangle\langle e_2|.$$

Clearly the conclusion of (b) holds when p is two dimensional, so let us assume that $p = |e\rangle\langle e|$ for some unit vector e. Since $\omega_{|e_1\rangle\langle e_1|}$ is absorbing we have

$$\lim \omega(\mathcal{T}_t(|e\rangle\langle e|)) = |\langle e, e_1\rangle|^2$$

for all states ω. When $|e\rangle\langle e| \neq |e_2\rangle\langle e_2|$ the above limit is strictly positive hence $\lim \omega(\mathcal{T}_t(|e\rangle\langle e|))^{1/t} = 1$ and the conclusion holds. Assume that $|e\rangle\langle e| = |e_2\rangle\langle e_2|$ and write $\omega = \omega_\rho$. We have

$$\min \sigma(\tilde{\mathcal{T}}_{*t}(I)) \leq \langle e_2, \tilde{\mathcal{T}}_{*t}(I)e_2\rangle = \langle e_2, \tilde{\mathcal{T}}_{*t}(|e_2\rangle\langle e_2|)e_2\rangle$$

hence

$$\lim \min \sigma(\tilde{\mathcal{T}}_{*t}(I))^{1/t} = \lim \langle e_2, \tilde{\mathcal{T}}_{*t}(I)e_2\rangle^{1/t} = e^{a-\eta}$$

by (34), and finally

$$\lim \omega_\rho(\mathcal{T}_t(|e_2\rangle\langle e_2|))^{1/t} = \lim \langle e_2, \tilde{\mathcal{T}}_{*t}(\rho)e_2\rangle^{1/t} = e^{a-\eta}$$

for all ρ strictly positive since in this case $cI \leq \rho \leq I$ for some $c > 0$. Assume now that $\rho = |f\rangle\langle f|$ for some unit vector f, and let h be a unit vector orthogonal to f. We have

$$\begin{aligned}\lim \langle e_2, \tilde{\mathcal{T}}_{*t}(I)e_2\rangle^{1/t} &= \max\{\limsup \langle e_2, \tilde{\mathcal{T}}_{*t}(|f\rangle\langle f|)e_2\rangle^{1/t}, \limsup \langle e_2, \tilde{\mathcal{T}}_{*t}(|h\rangle\langle h|)e_2\rangle^{1/t}\} \\ &= e^{a-\eta}\end{aligned}$$

hence

$$\begin{aligned}\liminf \min \sigma(\tilde{\mathcal{T}}_{*t}(|f\rangle\langle f|))^{1/t} &\leq \liminf \langle e_2, \tilde{\mathcal{T}}_{*t}(|f\rangle\langle f|)e_2\rangle^{1/t} \\ &\leq \limsup \langle e_2, \tilde{\mathcal{T}}_{*t}(|f\rangle\langle f|)e_2\rangle^{1/t} \leq e^{a-\eta},\end{aligned}$$

and since $\lim \min \sigma(\tilde{\mathcal{T}}_{*t}(|f\rangle\langle f|))^{1/t} = e^{a-\eta}$ by (24) we get $\lim \langle e_2, \tilde{\mathcal{T}}_{*t}(|f\rangle\langle f|)e_2\rangle^{1/t} = e^{a-\eta}$. □

The existence of an absorbing pure state can be seen as some uniform (with respect to the initial state) large deviation principle, as establishes the following corollary (the implication $(ii) \Rightarrow (i)$ is a direct consequence of Proposition 3).

Corollary 2 *For any quantum Markov semigroup* $(\mathcal{T}_t)$ *on* M_2 *the following statements are equivalent.*

- *(i)* $(\mathcal{T}_t)$ *admits an absorbing pure state;*
- *(ii) There exists a function* J *on the pure state space vanishing at a unique point such that for each state* ω *distinct from this point, the net of orthogonal measures representing* $(\mathcal{T}_{*t}(\omega))$ *satisfies a large deviation principle with powers* $(1/t)$ *and rate function* J.

When this holds the absorbing state is the point where J *vanishes.*

Remark 1 In [5] we defined a noncommutative large deviation principle for any net of states on any C^*-algebra A, where all the basic ingredients of the classical theory are replaced by their noncommutative counterparts, using the framework of noncommutative topology. Namely, open (resp. closed) sets are replaced by open (resp. closed) projections living in A^{**}, and the rate function J by a rate operator (more exactly, in order to avoid possibly infinite-valued operator we use instead the bounded upper semi-continuous operator as the counterpart of $\exp -J$, belonging also to A^{**}). Since $A = A^{**}$ when A is finite dimensional, all self-adjoint operators in A are continuous, in particular all projections are clopen. In this simple case, by definition, a net (ω_t) of states is said to satisfy a noncommutative large deviation principle with governing operator z if

$$\lim \omega_t(p)^{1/t} = \sup\{\lambda \in \sigma(z) : pE^z_{\{\lambda\}} \neq 0\} \qquad \text{for all projections } p \in A,$$

where $\sigma(z)$ and $E^z_{\{\lambda\}}$ denotes respectively the spectrum of z and the eigenspace corresponding to the eigenvalue λ (and $\inf\emptyset = +\infty$ by convention). The R.H.S. of the above equality can be written in the symbolic form " $\sup_p e^{-z}$" since it is the exact noncommutative version of $\sup_Y e^{-J}$ for Y open or closed ([5], Theorem 4.2). It follows that part (*b*) of Theorem 2 amounts to say that for each state $\omega \neq \omega_{|e_1\rangle\langle e_1|}$ the net of states $(\mathcal{T}_{*t}(\omega))_{t\geq 0}$ satisfies a noncommutative large deviation principle with governing operator $\exp -\mathcal{J}^*(|e_1\rangle\langle e_1|)$.

Remark 2 There are situations not covered by Theorem 4 for which large deviations, when they hold, are trivial. There are those where there is a faithful absorbing state ω_∞; we distinguish two cases.

(a) $\omega_\infty \neq \frac{1}{2}I$. By Lemma 2 the hypothesis of Proposition 2 (*a*) hold, hence (taking $r = 2$) the associated net of orthogonal measures satisfies a large deviation principle with a rate function vanishing at the two points given by the eigenvectors of ω_∞, and infinite-valued elsewhere.

(b) When $\omega_\infty = \frac{1}{2}I$ the large deviation is equivalent to the convergence of the one-dimensional projections given by eigenvectors, in which case the rate function has the same form as above. The "if" part of this assertion as well as the form of the rate function follow from Proposition 2. Conversely, if a large deviation principle holds for some net of representing measures, then necessarily the rate function vanishes on some point, say $\omega_{|e_1\rangle\langle e_1|}$. Varadhan's theorem implies

$$\lim \mu_t^{1/t}(\widehat{|e_1\rangle\langle e_1|}^{\,t}) = \max_{1\leq j\leq 2}\{\limsup a_{j,t}^{1/t}|\langle e_1, e_{j,t}\rangle|^2\}$$

$$= \sup_{u\in\mathcal{H},\|u\|=1} |\langle u, e_1\rangle|^2 e^{-J(\omega_{|u\rangle\langle u|})} = 1.$$

Since $\lim a_{1,t}^{1/t} = \lim a_{2,t}^{1/t} = 1$ we have $\lim|\langle e_1, e_{j,t}\rangle| = 1$ for some j (say $j = 1$) so that $\lim |e_{1,t}\rangle\langle e_{1,t}| = |e_1\rangle\langle e_1|$. Since

$$\lim\sum_{j=1}^{2} a_{j,t}|\langle e_1, e_{j,t}\rangle|^2 = \omega_\infty(|e_1\rangle\langle e_1|) = \frac{1}{2}$$

with $\lim a_{1,t} = \lim a_{2,t} = \frac{1}{2}$ we conclude that $\lim|\langle e_1, e_{2,t}\rangle| = 0$ hence $\lim|e_{2,t}\rangle\langle e_{2,t}| = |e_2\rangle\langle e_2|$ for some unit vector e_2 orthogonal to e_1.

4 Examples

In this section, we study a class of quantum Markov semigroups on M_2 arising in a special instance of the weak coupling limit ([12], [15], [9], [3]). We show that each of these semigroups has an absorbing state, so that large deviations follow from Theorem 4 when this state is pure. Before to describe the model we first recall briefly how works the weak coupling limit theory in this particular case ([2], [11]).

4.1 Weak coupling limit and squeezed-vacuum state

A quantum system with underlying Hilbert space $\mathcal{H}_0$ is coupled with the bosonic reservoir on some Hilbert space $\mathcal{H}_1$ (we shall assume $\mathcal{H}_1 = L^2(\mathbb{R}^d)$ to simplify) where some reference state ϕ on the associated CCR algebra is given. We will consider a special case where ϕ is a so-called *squeezed-vacuum state*, which can be defined as follows. For each pair of reals r, s let $T_{r,s}$ be the operator on $\mathcal{H}_1$ defined by

$$\forall f \in \mathcal{H}_1, \qquad T_{r,s}(f) = (\cosh r)f - \exp(-2is)(\sinh r)\overline{f}.$$

Then $T_{r,s}$ is real linear, invertible, and so induces a unique $*$-automorphism of the CCR algebra, defined on the Weyl operators by $W(f) \mapsto W(T_{r,s}f)$. The vacuum state is transformed by the above automorphism into the state $\phi_{r,s}$ defined by

$$\forall f \in \mathcal{H}_1, \qquad \phi_{r,s}(W(f)) = \exp(-\frac{1}{2}||T_{r,s}f||^2).$$

By means of the GNS representation of the CCR algebra with state $\phi_{r,s}$, we obtain for all $f \in \mathcal{H}_1$ a strongly continuous unitary group $(W_{r,s}(f))_{t\in\mathbb{R}}$; let $B_{r,s}(f)$ denote its infinitesimal generator. By definition such a $\phi_{r,s}$ is a *squeezed-vacuum* state if the following conditions hold.

- $\phi_{r,s}(B_{r,s}(f)) = 0$ for all $f \in \mathcal{H}_1$;
- There exists $f \in \mathcal{H}_1$ such that

  $$\phi_{r,s}(|B_{r,s}(f)|^2) \neq \phi_{r,s}(|B_{r,s}(if)|^2)$$

  and

  $$\phi_{r,s}(|B_{r,s}(f)|^2) \cdot \phi_{r,s}(|B_{r,s}(if)|^2) = \frac{1}{4}|\phi_{r,s}([B_{r,s}(f), B_{r,s}(if)])|.$$

In the sequel we assume that ϕ is such a squeezed vacuum state, and the corresponding $B_{r,s}$ is simply denoted by B; we also fix some $g \in \mathcal{H}_1 \backslash \{0\}$ and some $\omega_0 > 0$. The evolution of the composite system is given by the Hamiltonian

$$H^\lambda = H_0 \otimes 1 + 1 \otimes H_1 + \lambda V,$$

where H_0 is the Hamiltonian of the system, H_1 is the Hamiltonian of the free evolution of the reservoir (i.e. the second quantization of the one-particle Hamiltonian), λ the coupling constant, and

$$V = i(D \otimes (\frac{1}{2}B(g) - iB(ig)) - D^\dagger \otimes (\frac{1}{2}B(g) + iB(ig))),$$

where D is a bounded operator on H_0 satisfying

$$\exp(itH_0)D\exp(-itH_0) = \exp(-i\omega_0 t)D$$

and for each u in a dense subset of $\mathcal{H}_0$

$$\sum_{n=1}^{\infty} \frac{|\langle u, D^n u\rangle|}{[n/2]!} < \infty.$$

We put

$$U_t^\lambda = \exp(itH^0)\exp(-itH^\lambda), \quad (t \in \mathbb{R}).$$

We moreover assume that the function $t \mapsto \exp(-i\omega_0 t)\langle f, S_t h\rangle$ is integrable on $\mathbb{R}$ for all f, h in a dense linear subspace of the domain of Q, where (S_t) is the unitary one-particle free evolution, and Q is a real linear operator on $\mathcal{H}_1$ satisfying

$$\forall f \in \mathcal{H}_1, \qquad \Re\langle f, Qf\rangle = \phi(|B(f)|^2).$$

Then, U^λ_{t/λ^2} converges (in some appropriate sense) to some unitary transformation $U(t)$ (solution of a quantum stochastic differential equation) which induces on the algebra $\mathcal{B}(\mathcal{H}_0)$ of all bounded operators on $\mathcal{H}_0$ a quantum Markov semigroup $(\mathcal{T}_t)$. More precisely, for each normal state ω on $\mathcal{B}(\mathcal{H}_0)$ and each $x \in \mathcal{B}(\mathcal{H}_0)$ we have

$$\lim_{\lambda\to 0}(\omega\otimes\phi){U^\lambda_{t/\lambda^2}}^\dagger(x\otimes 1)U^\lambda_{t/\lambda^2} = \omega(\mathcal{T}_t(x)). \qquad (36)$$

The generator of $(\mathcal{T}_t)$ is obtained in terms of D and some parameters depending on ϕ, g, ω_0.

4.2 The model

We consider a two-level system so that $\mathcal{H}_0 = \mathbb{C}_2$, and we choose $D = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. In this case, the generator $\tilde{\mathcal{L}}_*$ associated to the predual semigroup arising as in (36) has the Lindblad form

$$\begin{aligned}\forall\rho\in M_2, \qquad \tilde{\mathcal{L}}_*(\rho) = {} & i\xi[DD^\dagger - D^\dagger D,\rho] - \tfrac{\nu+\eta}{2}(D^\dagger D\rho - 2D\rho D^\dagger + \rho D^\dagger D) \\ & -\tfrac{\nu}{2}(DD^\dagger\rho - 2D^\dagger\rho D + \rho DD^\dagger) + \overline{\zeta}D\rho D + \zeta D^\dagger\rho D^\dagger,\end{aligned}$$

where $\zeta\in\mathbb{C}$ and η,ν,ξ are reals satisfying $\eta>0$, $\nu\geq 0$ and

$$|\zeta|^2 \leq \nu(\nu+\eta). \qquad (37)$$

These parameters depend on the (implicitly fixed) choice of ϕ, g, ω_0, as described in 4.1; in particular,

$$\eta = \int_{-\infty}^{+\infty} \exp(-i\omega_0 t)\langle g, S_t g\rangle. \qquad (38)$$

Since our large deviation results will occur when $\nu = 0$ (equality which will be expressed in terms of η) and the other ones will not play any role, we do not give them here and refer to [11].

In order to show that each of these semigroups has an absorbing state ω_∞, we will use the Pauli matrices $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Recall that any self-adjoint operator $\rho\in M_2$ can be written in a unique way as $\rho = \frac{1}{2}(\mathrm{tr}\,\rho I + u(\rho)\cdot\sigma)$ where $u(\rho) = (u_1(\rho), u_2(\rho), u_3(\rho))$ with $u_i(\rho) = \mathrm{tr}\,\rho\sigma_i$ for $i\in\{1,2,3\}$, $\sigma = (\sigma_1,\sigma_2,\sigma_3)$, and $u(\rho)\cdot\sigma$ denotes the product $\sum_{i=1}^3 u_i(\rho)\sigma_i$. The diagonal form of ρ is given by

$$\rho = \frac{1}{2}(1+||u(\rho)||)p_{1,\rho} + \frac{1}{2}(1-||u(\rho)||)p_{2,\rho},$$

where $p_{1,\rho} = \frac{1}{2}(I + \frac{u(\rho)}{||u(\rho)||}\cdot\sigma)$ and $p_{2,\rho} = \frac{1}{2}(I - \frac{u(\rho)}{||u(\rho)||}\cdot\sigma)$ are the projections on the one-dimensional eigenspaces. Note that for each real a and each self-adjoint operator $x\in M_2$, $\mathrm{tr}\frac{1}{2}(I+u(\rho)\cdot\sigma)(aI+u(x)\cdot\sigma) = a + u(\rho)\cdot u(x)$.

Lemma 3 *For each positive trace-one operator $\rho\in M_2$ we put $\rho_t = \tilde{\mathcal{T}}_{*t}(\rho)$ for all $t\geq 0$.*

(a) $u_3(\rho_t) = e^{-(2\nu+\eta)t}(u_3(\rho) + \frac{\eta}{2\nu+\eta}) - \frac{\eta}{2\nu+\eta}$

(b) If $|\zeta|^2 - 4\xi^2 > 0$*, then there exist constants* $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ *such that*

- $u_1(\rho_t) = e^{m_1 t}(a_1 u_1(\rho) + b_1 u_2(\rho)) + e^{m_2 t}(a_2 u_1(\rho) + b_2 u_2(\rho))$.
- $u_2(\rho_t) = e^{m_1 t}(c_1 u_1(\rho) + d_1 u_2(\rho)) + e^{m_2 t}(c_2 u_1(\rho) + d_2 u_2(\rho))$.

where $m_1 = -(\nu + \frac{\eta}{2}) + \sqrt{|\zeta|^2 - 4\xi^2}$*,* $m_2 = -(\nu + \frac{\eta}{2}) - \sqrt{|\zeta|^2 - 4\xi^2}$.

(c) If $|\zeta|^2 - 4\xi^2 < 0$*, then there exist constants* $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ *such that*

- $u_1(\rho_t) = e^{mt}((a_1 u_1(\rho) + b_1 u_2(\rho)) \cos t\sqrt{4\xi^2 - |\zeta|^2}) + (a_2 u_1(\rho) + b_2 u_2(\rho)) \sin t\sqrt{4\xi^2 - |\zeta|^2})$
- $u_2(\rho_t) = e^{mt}((c_1 u_1(\rho) + d_1 u_2(\rho)) \cos t\sqrt{4\xi^2 - |\zeta|^2}) + (c_2 u_1(\rho) + d_2 u_2(\rho)) \sin t\sqrt{4\xi^2 - |\zeta|^2})$

where $m = -(\nu + \frac{\eta}{2})$.

(d) If $|\zeta|^2 - 4\xi^2 = 0$*, then there exist constants* $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ *such that*

- $u_1(\rho_t) = e^{2mt}((a_1 u_1(\rho) + b_1 u_2(\rho)) + t(a_2 u_1(\rho) + b_2 u_2(\rho)))$.
- $u_2(\rho_t) = e^{2mt}((c_1 u_1(\rho) + d_1 u_2(\rho)) + t(c_2 u_1(\rho) + d_2 u_2(\rho)))$.

where $m = -(\nu + \frac{\eta}{2})$.

Proof. Direct calculations yield

(i) $\tilde{\mathcal{L}}_*(I) = -\eta\sigma_3$

(ii) $\tilde{\mathcal{L}}_*(\sigma_1) = -(\nu + \frac{\eta}{2} - \Re\zeta)\sigma_1 + (2\xi - \Im\zeta)\sigma_2$,

(iii) $\tilde{\mathcal{L}}_*(\sigma_2) = -(2\xi + \Im\zeta)\sigma_1 - (\nu + \frac{\eta}{2} + \Re\zeta)\sigma_2$,

(iv) $\tilde{\mathcal{L}}_*(\sigma_3) = -(2\nu + \eta)\sigma_3$.

Differentiating $\tilde{\mathcal{T}}_{*t} = e^{t\tilde{\mathcal{L}}_*}$ yields $\frac{d\tilde{\mathcal{T}}_{*t}}{dt} = \tilde{\mathcal{L}}_* \circ \tilde{\mathcal{T}}_{*t}$, hence the following system of differential equations

- $\frac{d\tilde{\mathcal{T}}_{*t}(I)}{dt} = -\eta\tilde{\mathcal{T}}_{*t}(\sigma_3)$
- $\frac{d\tilde{\mathcal{T}}_{*t}(\sigma_1)}{dt} = -(\nu + \frac{\eta}{2} - \Re\zeta)\tilde{\mathcal{T}}_{*t}(\sigma_1) + (2\xi - \Im\zeta)\tilde{\mathcal{T}}_{*t}(\sigma_2)$
- $\frac{d\tilde{\mathcal{T}}_{*t}(\sigma_2)}{dt} = -(2\xi + \Im\zeta)\tilde{\mathcal{T}}_{*t}(\sigma_1) - (\nu + \frac{\eta}{2} + \Re\zeta)\tilde{\mathcal{T}}_{*t}(\sigma_2)$
- $\frac{d\tilde{\mathcal{T}}_{*t}(\sigma_3)}{dt} = -(2\nu + \eta)\tilde{\mathcal{T}}_{*t}(\sigma_3)$.

with initial conditions $\tilde{\mathcal{T}}_{*0}(I) = I$, $\tilde{\mathcal{T}}_{*0}(\sigma_1) = \sigma_1$, $\tilde{\mathcal{T}}_{*0}(\sigma_2) = \sigma_2$, $\tilde{\mathcal{T}}_{*0}(\sigma_3) = \sigma_3$. We then obtain

$$\tilde{\mathcal{T}}_{*t}(I) = I + \frac{\eta}{2\nu + \eta}(e^{-(2\nu+\eta)t} - 1)\sigma_3.$$

$$\tilde{\mathcal{T}}_{*t}(\sigma_3) = e^{-(2\nu+\eta)t}\sigma_3,$$

which gives (a). Put $x(t) = \tilde{\mathcal{T}}_{*t}(\sigma_1)$, $y(t) = \tilde{\mathcal{T}}_{*t}(\sigma_2)$. Clearly, $x(t)$ has only non-zero components $x_1(t)$ on σ_1 and $x_2(t)$ on σ_2; similarly, $y(t) = (y_1(t), y_2(t))$. The characteristic equation associated to the matrix given by the system

$$\begin{cases} x'(t) = -(\nu + \frac{\eta}{2} - \Re\zeta)x(t) + (2\xi - \Im\zeta)y(t) \\ \\ y'(t) = -(2\xi + \Im\zeta)x(t) - (\nu + \frac{\eta}{2} + \Re\zeta)y(t) \end{cases} \tag{39}$$

with initial conditions $x(0) = (\sigma_1, 0)$, $y(0) = (0, \sigma_2)$, is on each component

$$X^2 + (2\nu + \eta)X + \frac{\eta^2}{2} + \nu^2 + \nu\eta + 4\xi^2 - |\zeta|^2 \tag{40}$$

and so $\Delta = 4(|\zeta|^2 - 4\xi^2)$.

Assume $|\zeta|^2 - 4\xi^2 > 0$. The solutions of (40) being $m_1 = -(\nu + \frac{\eta}{2}) + \sqrt{|\zeta|^2 - 4\xi^2}$ and $m_2 = -(\nu + \frac{\eta}{2}) - \sqrt{|\zeta|^2 - 4\xi^2}$, the general solution of (39) is

$$\begin{cases} x_1(t) = a_1 e^{m_1 t} + a_2 e^{m_2 t}, & x_2(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t} \\ y_1(t) = b_1 e^{m_1 t} + b_2 e^{m_2 t}, & y_2(t) = d_1 e^{m_1 t} + d_2 e^{m_2 t} \end{cases}$$

for suitable constants $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$. It follows that

$$\tilde{\mathcal{T}}_{*t}(\sigma_1) = (a_1 e^{m_1 t} + a_2 e^{m_2 t})\sigma_1 + (c_1 e^{m_1 t} + c_2 e^{m_2 t})\sigma_2,$$

$$\tilde{\mathcal{T}}_{*t}(\sigma_2) = (b_1 e^{m_1 t} + b_2 e^{m_2 t})\sigma_1 + (d_1 e^{m_1 t} + d_2 e^{m_2 t})\sigma_2,$$

which gives (b).

Assume $|\zeta|^2 - 4\xi^2 < 0$. Then $m_1 = -(\nu + \frac{\eta}{2}) + i\sqrt{4\xi^2 - |\zeta|^2}$, $m_2 = -(\nu + \frac{\eta}{2}) - i\sqrt{4\xi^2 - |\zeta|^2}$, and the general solution of (39) is

$$\begin{cases} x_1(t) = e^{mt}(a_1 \cos t\sqrt{4\xi^2 - |\zeta|^2} + a_2 \sin t\sqrt{4\xi^2 - |\zeta|^2}) \\ y_1(t) = e^{mt}(b_1 \cos \sqrt{4\xi^2 - |\zeta|^2} + b_2 \sin \sqrt{4\xi^2 - |\zeta|^2}) \end{cases}$$

$$\begin{cases} x_2(t) = e^{mt}(c_1 \cos t\sqrt{4\xi^2 - |\zeta|^2} + c_2 \sin t\sqrt{4\xi^2 - |\zeta|^2}) \\ y_2(t) = e^{mt}(d_1 \cos \sqrt{4\xi^2 - |\zeta|^2} + d_2 \sin \sqrt{4\xi^2 - |\zeta|^2}) \end{cases}$$

where $m = -(\nu + \frac{\eta}{2})$, and $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ are suitable constants. We then obtain

$$\begin{aligned} \tilde{\mathcal{T}}_{*t}(\sigma_1) = e^{mt}(a_1 \cos t\sqrt{4\xi^2 - |\zeta|^2} + a_2 \sin t\sqrt{4\xi^2 - |\zeta|^2})\sigma_1 + \\ e^{mt}(c_1 \cos t\sqrt{4\xi^2 - |\zeta|^2} + c_2 \sin t\sqrt{4\xi^2 - |\zeta|^2})\sigma_2. \\ \tilde{\mathcal{T}}_{*t}(\sigma_2) = e^{mt}(b_1 \cos t\sqrt{4\xi^2 - |\zeta|^2} + b_2 \sin t\sqrt{4\xi^2 - |\zeta|^2})\sigma_1 + \\ e^{mt}(d_1 \cos t\sqrt{4\xi^2 - |\zeta|^2} + d_2 \sin t\sqrt{4\xi^2 - |\zeta|^2})\sigma_2, \end{aligned}$$

and (c) follows easily.

If $|\zeta|^2 - 4\xi^2 = 0$, then the unique solution of (40) is $2m$ with $m = -(\nu + \frac{\eta}{2})$, and the general solution of (39) is

$$\begin{cases} x_1(t) = e^{2mt}(a_1 + a_2 t), & x_2(t) = e^{mt}(c_1 + c_2 t) \\ y_1(t) = e^{2mt}(b_1 + b_2 t), & y_2(t) = e^{mt}(d_1 + d_2 t) \end{cases}$$

for suitable constants $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$, so that

$$\tilde{\mathcal{T}}_{*t}(\sigma_1) = e^{2mt}(a_1 + a_2 t)\sigma_1 + e^{2mt}(c_1 + c_2 t)\sigma_2$$

$$\tilde{\mathcal{T}}_{*t}(\sigma_2) = e^{2mt}(b_1 + b_2 t)\sigma_1 + e^{2mt}(d_1 + d_2 t)\sigma_2,$$

which imply (d). $\square$

Proposition 4 *The state* ω_∞ *given by the operator* $\rho_\infty = \frac{1}{2}(I - \frac{\eta}{(2\nu+\eta)}\sigma_3)$ *is absorbing for the semigroup* $(\mathcal{T}_t)_{t\geq 0}$.

Proof. Let ω_ρ be a state and put $\rho_t = \tilde{\mathcal{T}}_{*t}(\rho)$ for all $t \geq 0$. The hypotheses $\eta > 0$, $|\zeta|^2 \leq \nu(\nu+\eta)$, $\nu \geq 0$ imply (with the notations of Lemma 3) $m_1 m_2 = \frac{\eta^2}{2} + \nu^2 + \nu\eta + 4\xi^2 - |\zeta|^2 > 0$, and so $m_1 < 0$, $m_2 < 0$, $m < 0$. It follows that $\lim u_1(\rho_t) = 0 = \lim u_2(\rho_t)$ and $\lim u_3(\rho_t) = \frac{-\eta}{(2\nu+\eta)}$, hence for any ρ initial, and any positive $x = aI + u(x)\cdot\sigma$ in M_2, we have

$$\lim \mathrm{tr}\rho_t x = \lim(a + u(\rho_t)\cdot u(x)) = (a - \frac{\eta}{2\nu+\eta}u_3(x)) = \mathrm{tr}\rho_\infty x.$$

□

The above proposition shows that ω_∞ is pure if and only if $\nu = 0$. As it is easily seen from the expressions given in [11], we have

$$\nu = 0 \quad \Longleftrightarrow \quad \eta = \frac{1}{2}\Re\int_{-\infty}^{+\infty} \exp(-i\omega_0 t)(\langle g, S_t Q(g)\rangle + \langle ig, S_t Q(ig)\rangle). \tag{41}$$

Note that this holds in particular when $Q = I$ (i.e. ϕ is the vacuum state). We have $\rho_\infty = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} (= |e_1\rangle\langle e_1|)$, and since $\zeta = 0$ by (37), the Lindblad form of the generator becomes

$$\rho \in M_2, \qquad \tilde{\mathcal{L}}_*(\rho) = i\xi[DD^\dagger - D^\dagger D, \rho] - \frac{\eta}{2}(D^\dagger D\rho - 2D\rho D^\dagger + \rho D^\dagger D),$$

or equivalently as in Theorem 1,

$$\forall \rho \in M_2, \qquad \tilde{\mathcal{L}}_*(\rho) = y\rho + \rho y^* + z\rho z^*,$$

with $y = \begin{pmatrix} -\frac{\eta}{2} - i\xi & 0 \\ 0 & i\xi \end{pmatrix}$ and $z = \begin{pmatrix} 0 & 0 \\ \sqrt{\eta} & 0 \end{pmatrix}$; in particular $\mathcal{J}^*(|e_1\rangle\langle e_1|) = \eta|e_2\rangle\langle e_2|$.

The following large deviation result follows from Theorem 4 and Corollary 1. It shows that the exponential asymptotic behavior of $(\mathcal{T}_{*t})$ is controlled by the parameter (38); moreover, it does not depend on the choice of the squeezed-vacuum state ϕ, provided that ϕ satisfies the condition of (41).

Proposition 5 *If* $\nu = 0$, *then for each initial state* $\omega \neq \omega_{\rho_\infty}$ *the following conclusions hold.*

(a) The net of orthogonal measures representing $(\mathcal{T}_{*t}(\omega))_{t\geq 0}$ *satisfies a large deviation principle with powers* $(1/t)$ *and rate function*

$$J(\omega_{|e\rangle\langle e|}) = \begin{cases} 0 & \text{if } |e\rangle\langle e| = \rho_\infty \\ \eta & \text{if } |e\rangle\langle e| = I - \rho_\infty \\ +\infty & \textit{otherwise.} \end{cases}$$

(b) For each projection $p \in M_2\backslash\{0\}$ *we have*

$$\lim \frac{1}{t}\log\omega(\mathcal{T}_t(p)) = \begin{cases} -\eta & \text{if } p = I - \rho_\infty \\ 0 & \textit{otherwise.} \end{cases}$$

Remark 3 The explicit expressions of $\mathcal{T}_{*t}(\omega)$ given by Lemma 3 (c) allow a direct proof of Proposition 5 (a). Indeed, easy calculations yield

$$u_1(\rho_t) = e^{-\frac{\eta}{2}t}(u_1(\rho)\cos(2\xi t) - u_2(\rho)\sin(2\xi t)),$$

$$u_2(\rho_t) = e^{-\frac{\eta}{2}t}(u_1(\rho)\sin(2\xi t) + u_2(\rho)\cos(2\xi t)),$$

so that

$$1 - ||u(\rho_t)||^2 = -e^{-2\eta t}(1 + u_3(\rho))^2 - e^{-\eta t}(u_1(\rho)^2 + u_2(\rho)^2 - 2(u_3(\rho) + 1)).$$

If $u_1(\rho)^2 + u_2(\rho)^2 - 2(u_3(\rho) + 1) = 0$, then necessarily $u_3(\rho) = -1$, $u_1(\rho) = u_2(\rho) = 0$, and $\rho = \rho_\infty$, which is excluded. It follows that $u_1(\rho)^2 + u_2(\rho)^2 - 2(u_3(\rho) + 1) < 0$ and $\lim(1 - ||u(\rho_t)||^2)^{1/t} = e^{-\eta}$. Since $\lim(1 + ||u(\rho_t)||)^{1/t} = 1$ and

$$e^{-\eta} = \lim(1 - ||u(\rho_t)||^2)^{1/t} = \limsup(1 - ||u(\rho_t)||)^{1/t} \lim(1 + ||u(\rho_t)||)^{1/t},$$

we get $\lim(1 - ||u(\rho_t)||)^{1/t} = e^{-\eta}$. Since $\rho_t = \frac{1}{2}(1 + ||u(\rho_t)||)p_{1,\rho_t} + \frac{1}{2}(1 - ||u(\rho_t)||)p_{2,\rho_t}$ the conclusion follows from Proposition 2.

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