Functional approach of large deviations in general spaces

Henri Comman[*]

Abstract

Let X be a topological space, (μ_α) a net of Borel probability measures on X, and (t_α) a net in $]0,\infty[$ converging to 0. Let $\mathcal{A}$ be a set of continuous functions such that for all $x \in X$ that can be suitably distinguished by some continuous functions from any closed set not containing x, $\mathcal{A}$ contains such a distinguishing function. Assuming that $\Lambda(h) = \log \lim(\int_X e^{h(x)/t_\alpha} \mu_\alpha(dx))^{t_\alpha}$ exists for all $h \in \mathcal{A}$, we give a sufficient condition in order that (μ_α) satisfies a large deviation principle with powers (t_α) and not necessary tight rate function. When X is completely regular (not necessary Hausdorff), this condition is also necessary, and so strictly weaker than exponential tightness; this allows us to strengthen Bryc's theorem in various ways. We give the general form of a rate function in terms of $\mathcal{A}$. A Prohorov-type theorem with a weaker notion than exponential tightness is obtained, which improves known results.

Key words: Large deviations; converse Varadhan's theorem problem.

1 Introduction

Let (μ_α) be a net of Borel probability measures on a topological space X, and (t_α) a net in $]0,\infty[$ converging to 0. For each $[-\infty,+\infty[$-valued measurable function h on X, we define $\Lambda(h) = \log \lim \mu_\alpha^{t_\alpha}(e^{h/t_\alpha})$ provided the limit exists in $[-\infty,+\infty]$, where $\mu_\alpha^{t_\alpha}(e^{h/t_\alpha})$ stands for $(\int_X e^{h(x)/t_\alpha} \mu_\alpha(dx))^{t_\alpha}$.

In order to prove a large deviation principle for $(\mu_\alpha^{t_\alpha})$, the so-called functional approach consists first to establish the existence of $\Lambda(\cdot)$ on some class of continuous functions, and next to look for a sufficient condition, either of set-theoretic type, or on the functional $\Lambda(\cdot)$. When X is normal Hausdorff and $\Lambda(\cdot)$ exists on the set $\mathcal{C}_b(X)$ of real-valued bounded continuous functions on X, such necessary and sufficient conditions have been given in [2]. When the existence of $\Lambda(\cdot)$ is not established on all $\mathcal{C}_b(X)$, the best known conditions involve a tightness condition: when X is completely regular Hausdorff and $\Lambda(\cdot)$ exists on some well-separating class, the exponential tightness is required (Bryc's theorem); when X is Polish and $\Lambda(\cdot)$ exists on the set of bounded Lipschitz continuous functions, it is required that $\Lambda(h) = \sup_{x\in X}\{h(x) - J(x)\}$ for all h in this set, where J is a tight rate function (which also implies exponential tightness) ([4]).

---

[*] Department of Mathematics, University of Santiago de Chile, Bernardo O'Higgins 3363, Santiago, Chile. E-mail: hcomman@usach.cl

In this paper, we will deal only with set-theoretic conditions; our aim is simultaneously to weaken the exponential tightness hypothesis and to reduce the set of continuous functions on which $\Lambda(\cdot)$ has to be defined in order to get large deviations with not necessary tight rate function; this is achieved in Theorem 3, with moreover no condition on X.

Indeed, since Varadhan's theorem is valid in general spaces without any hypothesis on the function giving the large deviations upper and lower bounds ([2], Corollary 3.4), it is natural to ask about the converse in a general setting. There is a lack in the literature about this problem when X is not completely regular Hausdorff (however, there are Hausdorff spaces which fail to be completely regular only because of one point). The fact is that the usual techniques are first heavily based on compactness arguments forcing the compact sets to be Borel sets (which is not necessarily the case if X is not Hausdorff), and next they require approximations of all open sets by elements of $\mathcal{C}_b(X)$ (which is not ensured if X is not completely regular).

We use here an alternative method valid for any topological space, which is based on the variational forms of the functionals $\underline{\Lambda}(\cdot) = \log\liminf \mu_\alpha^{t_\alpha}(e^{\cdot/t_\alpha})$ and $\overline{\Lambda}(\cdot) = \log\limsup \mu_\alpha^{t_\alpha}(e^{\cdot/t_\alpha})$. It leads to a condition for large deviations, only assuming the existence of $\Lambda(\cdot)$ on a class of continuous functions which suitably approximates what could be called the "completely regular part" of X.

More precisely, let X_0 denote the set of points $x \in X$ such that x can be distinguished from any closed set F not containing x, in the sense that for each real $s > 0$, and each real $t > 0$, there exists some continuous functions h such that $h(x) \geq -t$ and $h_{|F} \leq -s$; we then define an approximating class as a class of functions containing such a distinguishing function for any such pair (x, F). We give a sufficient condition only in terms of the set-function $\limsup \mu_\alpha^{t_\alpha}(\cdot)$ with domain the closed sets of X, in order that (μ_α) satisfies a large deviation principle with powers (t_α) and not necessary tight rate function, assuming that $\Lambda(\cdot)$ exists on some approximating class. The definition of such a class implying no hypothesis on X, this condition works for any topological space; it is moreover necessary, and so strictly weaker than exponential tightness, when X is completely regular (not necessarily Hausdorff); in this case, various classical results are strengthened, and in particular Bryc's theorem (Corollary 4); a more practical sufficient condition, involving the open covers of X is obtained (Corollary 4); it generalizes the notion of τ-smoothness for a net of measures and leads to Prohorov-type theorems with exponential τ-smoothness in place of exponential tightness (Corollary 5); also, the general form of a rate function is given in terms of any approximating class (Corollary 2). We emphasize that no compactness arguments are used, the open cover property being replaced by conditions on the net $(\mu_\alpha^{t_\alpha})$.

The paper is organized as follows. In Section 2, we establish a variational formula for $\underline{\Lambda}(h)$, valid for any $[-\infty, +\infty[$-valued measurable function h satisfying the tail condition of Varadhan's theorem (Theorem 2); together with the formula for $\overline{\Lambda}(h)$ given in [2], this leads to a sufficient condition for the existence of $\Lambda(h)$, for a given h (Corollary 1).

In Section 3, we introduce the notion of approximating class (say $\mathcal{A}$); we define a function $l_{\mathcal{A}}$ in terms of this class, and give conditions in order that large deviation lower (resp. upper) bounds hold with the function $l_{\mathcal{A}}$, assuming the existence of $\Lambda(\cdot)$ on $\mathcal{A}$ (Proposition 2 and Proposition 3). When all the above conditions hold, we prove that $l_{\mathcal{A}}$ is lower semi-continuous, and that these conditions are also necessary when X is completely regular (Theorem 3); this case gives rise to various corollaries which improve known results.

1.1 Notations

Throughout the paper, X is a topological space without others hypotheses than those explicitly stated, in particular X need not satisfy any separation axiom. The Hausdorff property is not included in the definitions of regular, completely regular, normal and compact; in particular, the points being not necessarily closed, normality does not imply regularity (in [2], the Hausdorff property is implicitly included in the definition of normality, although it is used only in the last section of the paper). Let $\mathcal{F}$ (resp. $\mathcal{G}$) denote the set of closed (resp. open) subsets of X. The closure and interior of a set $Y \subset X$ are respectively denoted by $\overline{Y}$ and $\overset{\circ}{Y}$. The L^∞-norm on the set of real-valued bounded Borel functions on X is denoted by $|| \cdot ||$.

By definition, (μ_α) satisfies a large deviation principle with powers (t_α) if there exists a $[0, +\infty]$-valued function l on X such that

$$\limsup \mu_\alpha^{t_\alpha}(F) \leq \sup_{x \in F} e^{-l}(x) \qquad \text{for all } F \in \mathcal{F} \tag{1}$$

and

$$\sup_{x \in G} e^{-l}(x) \leq \liminf \mu_\alpha^{t_\alpha}(G) \qquad \text{for all } G \in \mathcal{G}. \tag{2}$$

The lower regularization $\overset{\circ}{l}$ of l (i.e. the greatest lower semi-continuous function lesser than l) is called a rate function for $(\mu_\alpha^{t_\alpha})$, which is said to be tight when it has compact level sets. When $\mathcal{F}$ in (1) is replaced by the set of compact subsets of X (in which case it is assumed that compact sets are Borel sets), we say that a vague large deviation principle holds. When (1) (resp. (2)) holds, we say that (μ_α) satisfies the large deviation upper (resp. lower) bounds with the function l.

2 Variational formula for $\underline{\Lambda}(\cdot)$

In [2], we have given a general variational form of $\overline{\Lambda}(\cdot)$, which yields to a criterion of existence of $\Lambda(\cdot)$ on $\mathcal{C}_b(X)$ when X is normal. We will need in the sequel a sufficient condition for the existence of $\Lambda(h)$ for a given $[-\infty, +\infty[$-valued bounded above continuous function h on X, with X general. In this section, we obtain a variational formula for $\underline{\Lambda}(\cdot)$, and get in Corollary 1 the desired condition, valid moreover for all $[-\infty, +\infty[$-valued Borel measurable functions on X satisfying the tail condition (3).

For each map $h : X \to [-\infty, +\infty[$ we put $F^h_{\lambda,\varepsilon} = \{x : \lambda - \varepsilon \leq e^{h(x)} \leq \lambda + \varepsilon\}$ and $G^h_{\lambda,\varepsilon} = \{x : \lambda - \varepsilon < e^{h(x)} < \lambda + \varepsilon\}$ for all $\lambda \geq 0$ and for all $\varepsilon > 0$. We denote by $\mathcal{C}^T(X)$ (resp. $\mathcal{B}^T(X)$) the set of $[-\infty, +\infty[$-valued continuous (resp. Borel measurable) functions on X satisfying

$$\lim_{M \to \infty} \limsup \mu_\alpha^{t_\alpha}(e^{h/t_\alpha} 1_{\{e^h > M\}}) = 0. \tag{3}$$

We first recall the following variational form of $\overline{\Lambda}(\cdot)$ obtained in [2].

Theorem 1 *For each $h \in \mathcal{B}^T(X)$, there exists $M \in]0, \infty[$ such that*

$$e^{\overline{\Lambda}(h)} = \sup_{\lambda \geq 0, \varepsilon > 0} \{(\lambda - \varepsilon) \limsup \mu_\alpha^{t_\alpha}(F^h_{\lambda,\varepsilon})\} = \sup_{\{x \in X, \varepsilon > 0 : \varepsilon < e^{h(x)} \leq M\}} \{(e^{h(x)} - \varepsilon) \limsup \mu_\alpha^{t_\alpha}(G^h_{e^{h(x)},\varepsilon})\}.$$

We give now the following analogue for $\underline{\Lambda}(\cdot)$.

Theorem 2 *For each $h \in \mathcal{B}^T(X)$, there exists $M \in]0,\infty[$ such that*

$$e^{\Lambda(h)} = \liminf \sup_{\lambda \geq 0, \varepsilon > 0} \{(\lambda - \varepsilon)\mu_\alpha^{t_\alpha}(F^h_{\lambda,\varepsilon})\} \tag{4}$$

$$= \lim_{\varepsilon \to 0} \liminf \sup_{\{0 < e^h \leq M\}} \{e^{h(x)}\mu_\alpha^{t_\alpha}(G^h_{e^{h(x)},\varepsilon})\}. \tag{5}$$

Proof. Put $g = e^h$, $G_{\lambda,\varepsilon} = G^h_{\lambda,\varepsilon}$, $F_{\lambda,\varepsilon} = F^h_{\lambda,\varepsilon}$ for all $\lambda \geq 0$ and for all $\varepsilon > 0$. For all $M \geq 0$, for all $N \in \mathbb{N}^*$ and for all $1 \leq j \leq N$, we define

$$F_{M,N,j} = \{x : (j-1)M/N \leq g(x) \leq jM/N\}.$$

We have

$$\mu_\alpha^{t_\alpha}(g^{1/t_\alpha}) \geq \mu_\alpha^{t_\alpha}(g^{1/t_\alpha}1_{F_{\lambda,\varepsilon}}) \geq (\lambda - \varepsilon)\mu_\alpha^{t_\alpha}(F_{\lambda,\varepsilon})$$

for all $\lambda \geq 0$ and for all $\varepsilon > 0$, and so

$$\liminf \mu_\alpha^{t_\alpha}(g^{1/t_\alpha}) \geq \liminf \sup_{\lambda \geq 0, \varepsilon > 0} \{(\lambda - \varepsilon)\mu_\alpha^{t_\alpha}(F_{\lambda,\varepsilon})\}$$

$$\geq \lim_{\varepsilon \to 0} \liminf \sup_{\{g \leq M\}} \{g(x)\mu_\alpha^{t_\alpha}(G_{g(x),\varepsilon})\},$$

for all $M \in]0,\infty[$. Thus, in order to prove (4) and (5), we have to prove that

$$\liminf \mu_\alpha^{t_\alpha}(g^{1/t_\alpha}) \leq \lim_{\varepsilon \to 0} \liminf \sup_{\{g \leq M\}} \{g(x)\mu_\alpha^{t_\alpha}(G_{g(x),\varepsilon})\} \tag{6}$$

for some $M \in]0,\infty[$. We have for each $M \in]0,\infty[$,

$$\liminf \mu_\alpha^{t_\alpha}(g^{1/t_\alpha}) \leq \liminf (\sum_{j=1}^{N} \mu_\alpha(g^{1/t_\alpha}1_{F_{M,N,j}}) + \mu_\alpha(g^{1/t_\alpha}1_{\{g>M\}}))^{t_\alpha}$$

$$\leq \liminf \{\max_{1 \leq j \leq N} \mu_\alpha^{t_\alpha}(g^{1/t_\alpha}1_{F_{M,N,j}}) \vee \mu_\alpha^{t_\alpha}(g^{1/t_\alpha}1_{\{g>M\}})\},$$

and so

$$\liminf \mu_\alpha^{t_\alpha}(g^{1/t_\alpha}) \leq \liminf \{\max_{1 \leq j \leq N} \|g1_{F_{M,N,j}}\|\mu_\alpha^{t_\alpha}(F_{M,N,j}) \vee \mu_\alpha^{t_\alpha}(g^{1/t_\alpha}1_{\{g>M\}})\}. \tag{7}$$

Let $M \to \infty$, $N \to \infty$ in (7) and use (3) to obtain some $M_0 \in]0,\infty[$ such that

$$\liminf \mu_\alpha^{t_\alpha}(g^{1/t_\alpha}) \leq \liminf_{N \to \infty} \liminf \max_{1 \leq j \leq N} \{\|g1_{F_{M_0,N,j}}\|\mu_\alpha^{t_\alpha}(F_{M_0,N,j})\}.$$

Thus, to obtain (6) it suffices to show that

$$\liminf_{N \to \infty} \liminf \max_{1 \leq j \leq N} \{\|g1_{F_{M_0,N,j}}\|\mu_\alpha^{t_\alpha}(F_{M_0,N,j})\}$$

$$\leq \lim_{\varepsilon \to 0} \liminf \sup_{\{g \leq M_0\}} \{g(x)\mu_\alpha^{t_\alpha}(G_{g(x),\varepsilon})\}. \tag{8}$$

For all $\nu > 0$, for all $0 < \varepsilon < \nu$, for all $\lambda \in [0, M_0]$, and for all $N > M_0/\varepsilon$ we have

$$\|g1_{F_{M_0,N,j_\lambda}}\|\mu_\alpha^{t_\alpha}(F_{M_0,N,j_\lambda}) \leq (\lambda + \nu)\mu_\alpha^{t_\alpha}(G_{\lambda,\varepsilon}) \tag{9}$$

where j_λ is such that $\lambda \in [(j_\lambda - 1)M_0/N, j_\lambda M_0/N]$ (since $[(j_\lambda - 1)M_0/N, j_\lambda M_0/N] \subset]\lambda - \varepsilon, \lambda + \varepsilon[$). When λ ranges over $[0, M_0]$, j_λ ranges over $\{j : 1 \leq j \leq N\}$, and (9) implies

$$\begin{aligned} &\max_{1 \leq j \leq N} \{\|g1_{F_{M_0,N,j}}\| \mu_\alpha^{t_\alpha}(F_{M_0,N,j})\} \\ &\leq \sup_{0 \leq \lambda \leq M_0} \{(\lambda + \nu)\mu_\alpha^{t_\alpha}(G_{\lambda,\varepsilon})\} \end{aligned} \tag{10}$$

for all $\varepsilon < \nu$ and for all $N > M_0/\varepsilon$. Notice that for all $N \in \mathbb{N}^*$ and for all $1 \leq j \leq N$, if $F_{M_0,N,j} \neq \emptyset$ then $j = j_{g(x)}$ for some $x \in X$. Thus, it suffices to consider $\lambda \in \{g(x) : x \in X, g(x) \leq M_0\}$ on the L.H.S. of (10), which then implies

$$\begin{aligned} &\liminf_{N\to\infty} \liminf \max_{1 \leq j \leq N} \{\|g1_{F_{M_0,N,j}}\| \mu_\alpha^{t_\alpha}(F_{M_0,N,j})\} \\ &\leq \liminf \sup_{\{g \leq M_0\}} \{(g(x) + \nu)\mu_\alpha^{t_\alpha}(G_{g(x),\varepsilon})\}. \end{aligned} \tag{11}$$

for all $\varepsilon < \nu$. Let first $\varepsilon \to 0$ and next $\nu \to 0$ in (11) to get (8). It follows that (6), (4) and (5) hold. $\square$

Corollary 1 *Let $h \in \mathcal{B}^T(X)$ (resp. $h \in \mathcal{C}^T(X)$). If $\gamma : \{F^h_{\lambda,\varepsilon} : \lambda \geq 0, \varepsilon > 0\} \cup \{G^h_{\lambda,\varepsilon} : \lambda \geq 0, \varepsilon > 0\} \to [0,1]$ is a map satisfying*

$$\limsup \mu_\alpha^{t_\alpha}(F^h_{\lambda,\delta}) \leq \gamma(F^h_{\lambda,\delta}) \leq \gamma(G^h_{\lambda,\varepsilon}) \leq \liminf \mu_\alpha^{t_\alpha}(G^h_{\lambda,\varepsilon}) \tag{12}$$

$$(\textit{resp.} \qquad \limsup \mu_\alpha^{t_\alpha}(\overset{\circ}{F^h_{\lambda,\delta}}) \leq \gamma(F^h_{\lambda,\delta}) \leq \gamma(G^h_{\lambda,\varepsilon}) \leq \liminf \mu_\alpha^{t_\alpha}(\overline{G^h_{\lambda,\varepsilon}})) \tag{13}$$

for all reals $\lambda > \varepsilon > \delta > 0$, then $\Lambda(h)$ exists and

$$e^{\Lambda(h)} = \sup_{\lambda \geq 0, \varepsilon > 0} \{(\lambda - \varepsilon)\gamma(F^h_{\lambda,\varepsilon})\} = \sup_{M \geq \lambda > \varepsilon > 0} \{(\lambda - \varepsilon)\gamma(G^h_{\lambda,\varepsilon})\} \tag{14}$$

for some real M.

Proof. Let $\gamma : \{F^h_{\lambda,\varepsilon} : \lambda \geq 0, \varepsilon > 0\} \cup \{G^h_{\lambda,\varepsilon} : \lambda \geq 0, \varepsilon > 0\} \to [0,1]$ satisfying (12) (resp. (13)). If $\Lambda(h)$ does not exist, then by Theorem 1 and Theorem 2, there exists $\lambda_0 > \varepsilon_0 > 0$ and $\nu > 0$ such that for all $\lambda \geq 0$ and for all $\varepsilon > 0$, we have

$$(\lambda_0 - \varepsilon_0) \limsup \mu_\alpha^{t_\alpha}(G^h_{\lambda_0,\varepsilon_0}) > \nu + (\lambda - \varepsilon) \liminf \mu_\alpha^{t_\alpha}(F^h_{\lambda,\varepsilon}), \tag{15}$$

and by taking $\lambda = \lambda_0$ and $\varepsilon_0 < \varepsilon < \min\{\varepsilon_0 + \nu, \lambda_0\}$ in (15), we get

$$(\lambda_0 - \varepsilon_0) \limsup \mu_\alpha^{t_\alpha}(G^h_{\lambda_0,\varepsilon_0}) > (\lambda_0 - \varepsilon_0) \liminf \mu_\alpha^{t_\alpha}(F^h_{\lambda_0,\varepsilon}),$$

which contradicts (12) (resp. (13), since $\overset{\circ}{F^h_{\lambda_0,\varepsilon_0}} \supset G^h_{\lambda_0,\varepsilon_0}$ and $F^h_{\lambda_0,\varepsilon} \supset \overline{G^h_{\lambda_0,\varepsilon}}$ if h is continuous). Thus $\Lambda(h)$ exists, and by Theorem 1 and Theorem 2 we have

$$\sup_{\lambda \geq 0, \varepsilon > 0} \{(\lambda - \varepsilon)\gamma(G^h_{\lambda,\varepsilon})\} \leq e^{\Lambda(h)} \leq \sup_{\lambda \geq 0, \varepsilon > 0} \{(\lambda - \varepsilon)\gamma(F^h_{\lambda,\varepsilon})\}.$$

Suppose that

$$\sup_{\lambda \geq 0, \varepsilon > 0} \{(\lambda - \varepsilon)\gamma(F^h_{\lambda,\varepsilon})\} > \sup_{\lambda \geq 0, \varepsilon > 0} \{(\lambda - \varepsilon)\gamma(G^h_{\lambda,\varepsilon})\} + \nu$$

for some $\nu > 0$. Then, there exists $\lambda_0 > \varepsilon_0 > 0$ such that

$$(\lambda_0 - \varepsilon_0)\gamma(F^h_{\lambda_0,\varepsilon_0}) > (\lambda_0 - \varepsilon + \nu)\gamma(G^h_{\lambda,\varepsilon}). \tag{16}$$

Take $\varepsilon_0 < \varepsilon < \min\{\varepsilon_0 + \nu, \lambda_0\}$ in (16), and obtain

$$\gamma(F^h_{\lambda_0,\varepsilon_0}) > \gamma(G^h_{\lambda_0,\varepsilon})$$

which contradicts (12) (resp. (13)), so that (14) holds. □

3 Approximating class, lower and upper bounds

In this section, we consider the set X_0 of points $x \in X$ such that x can be distinguished from any closed set F not containing x, in the sense of (17); an approximating class is then a class of functions distinguishing any such pair (x, F). Minimal requirements are made, in particular an approximating class need not contain the constants, neither be stable by finite minima, nor satisfy the two-points property. When X is completely regular (i.e., $X_0 = X$ by Proposition 1), each set of (x, F)-Urysohn maps (where F ranges over $\mathcal{F}$ and x over $X\backslash F$) gives rise to such a class. For any approximating class $\mathcal{A}$, we define a function $l_{\mathcal{A}}$ in terms of this class, and give conditions in order that large deviation lower (resp. upper) bounds hold with the function $l_{\mathcal{A}}$ (Proposition 2 and Proposition 3).

Definition 1 Let X_0 be the set of points x of X such that for each $G \in \mathcal{G}$ containing x, each real $s > 0$, and each real $t > 0$, there exists $h \in \mathcal{C}_b(X)$ such that

$$e^{-t} 1_{\{x\}} \leq e^h \leq 1_G \vee e^{-s}. \tag{17}$$

A class $\mathcal{A}$ of $[-\infty, +\infty[$-valued continuous functions on X is said to be *approximating* if for each $x \in X_0$, each $G \in \mathcal{G}$ containing x, each real $s > 0$, and each real $t > 0$, $\mathcal{A}$ contains some function satisfying (17). We then put

$$-l_{\mathcal{A}}(x) = \sup_{t>0} \inf_{\{h \in \mathcal{A} \cap \mathcal{C}^T(X) : h(x) \geq -t\}} \overline{\Lambda}(h) \qquad \text{for all } x \in X_0,$$

$$-l_{\mathcal{A}}(x) = \inf_{G \supset \{x\}, G \in \mathcal{G}} \inf_{0<s<\infty} \sup_{\mathcal{A}_{G,s}} \overline{\Lambda}(h) \qquad \text{for all } x \in X\backslash X_0,$$

where

$$\mathcal{A}_{G,s} = \bigcup_{x \in G \cap X_0, t>0} \{h \in \mathcal{A} : e^h \leq 1_G \vee e^{-s}, h(x) \geq -t\} \quad \text{for all } G \in \mathcal{G} \text{ and } s \in]0, +\infty[.$$

Remark 1 Let $\mathcal{A}$ be an approximating class. For each $G \in \mathcal{G}$, we have $\mathcal{A}_{G,s} = \emptyset$ for some real $s > 0$ if and only if $\mathcal{A}_{G,s} = \emptyset$ for all reals $s > 0$ if and only if $G \subset X\backslash X_0$; in particular, $l_{\mathcal{A}}(x) = +\infty$ for all $x \in X\backslash\overline{X_0}$.

Although the definition of $\mathcal{A}$ does not imply any separation axiom for X, we have however the following.

Proposition 1 *X is completely regular if and only if $X_0 = X$.*

Proof. Let X be completely regular. For each $G \in \mathcal{G}$ and each $x \in G$, there exists $h_{G,x} \in \mathcal{C}_b(X)$ such that $1_{\{x\}} \leq h_{G,x} \leq 1_G$. Then, $1_{\{x\}} \leq e^{sh_{G,x}-s} \leq 1_G \vee e^{-s}$ for each real $s > 0$, so that the set $\{sh_{G,x} - s : G \in \mathcal{G}, x \in G, s \in]0,\infty[\}$ is an approximating class in $\mathcal{C}_b(X)$, whence $X_0 = X$. Suppose that $X = X_0$, and let $\mathcal{A}$ be an approximating class. For each $G \in \mathcal{G}$, each $x \in G$ and each $t > 0$, let $h_{G,x,s,t} \in \mathcal{A}$ such that $e^{-t}1_{\{x\}} \leq e^{h_{G,x,s,t}} \leq 1_G \vee e^{-s}$. Then, $\{((h_{G,x,2,1} \vee -2) \wedge -1) + 2 : G \in \mathcal{G}, x \in G\}$ is a set of $[0,1]$-valued functions in $\mathcal{C}_b(X)$, which distinguishes points and closed sets, so that X is completely regular ([5], pp. 117). $\square$

Example 1 If X is regular normal and second countable, then X has a countable approximating class: let $\mathcal{G}_0$ be a countable basis of the topology of X; for each $G \in \mathcal{G}_0$ and $x \in G$, there exists $G' \in \mathcal{G}_0$ satisfying $x \in G' \subset \overline{G'} \subset G$, and the set $\{nh_{G,\overline{G'}} - n : h_{G,\overline{G'}} \in \mathcal{C}_b(X), 1_{\overline{G'}} \leq h_{G,\overline{G'}} \leq 1_G, G \in \mathcal{G}_0, G' \in \mathcal{G}_0, \overline{G'} \subset G, n \in \mathbb{N}\}$ makes the job.

Example 2 If X is metric with metric d, then the set $\{h_{G,x,s}; G \in \mathcal{G}, x \in G\}$ where $h_{G,x,s}(y) = -s(\frac{d(x,y)}{d(x,G^c)} \wedge 1)$ for all $y \in X$, is an approximating class of bounded Lipschitz continuous functions.

The existence of $\Lambda(\cdot)$ on an approximating class $\mathcal{A}$ implies large deviation lower bounds with the function $l_{\mathcal{A}}$, as shows the following.

Proposition 2 *For each approximating class $\mathcal{A}$, we have*

$$-\inf_{x \in G} l_{\mathcal{A}}(x) \leq \inf_{0<s<\infty} \sup_{\mathcal{A}_{G,s}} \overline{\Lambda}(h) \qquad \text{for all } G \in \mathcal{G}. \tag{18}$$

If $\Lambda(h)$ exists for all $h \in \mathcal{A}$, then

$$-\inf_{x \in G} l_{\mathcal{A}}(x) \leq \inf_{0<s<\infty} \sup_{\mathcal{A}_{G,s}} \Lambda(h) \leq \liminf t_\alpha \log \mu_\alpha(G) \qquad \text{for all } G \in \mathcal{G}. \tag{19}$$

Proof. If

$$-l_{\mathcal{A}}(x) > \inf_{0<s<\infty} \sup_{h \in \mathcal{A}_{G,s}} \overline{\Lambda}(h)$$

for some $G \in \mathcal{G}$ and some $x \in G$, then $x \in X_0$ and there exists some reals $s_0 > 0$ and $t_0 > 0$ such that

$$\inf_{\{h \in \mathcal{A} \cap \mathcal{C}^T(X) : h(x) \geq -t_0\}} \overline{\Lambda}(h) > \sup_{h \in \mathcal{A}_{G,s_0}} \overline{\Lambda}(h) \tag{20}$$

Since $\mathcal{A}$ is approximating, $\{h \in \mathcal{A}_{G,s_0} : h(x) \geq -t_0\}$ is nonempty, and (20) gives the contradiction. Therefore, (18) holds. Suppose that $\Lambda(h)$ exists for all $h \in \mathcal{A}$. Let $G \in \mathcal{G}$. If $G \subset X \backslash X_0$, then (19) holds obviously. Suppose that $G \not\subset X \backslash X_0$. If

$$\nu < \inf_{0<s<\infty} \sup_{h \in \mathcal{A}_{G,s}} e^{\Lambda(h)}$$

for some real ν, then for each real s there exists $h_s \in \mathcal{A}_{G,s}$ such that

$$\nu < e^{\Lambda(h_s)} \leq e^{-s} + \liminf \mu_\alpha^{t_\alpha}(G).$$

When $s \to +\infty$ we obtain

$$\nu \leq \liminf \mu_\alpha^{t_\alpha}(G),$$

and so

$$\liminf \mu_\alpha^{t_\alpha}(G) \geq \inf_{0<s<\infty} \sup_{h\in\mathcal{A}_{G,s}} e^{\Lambda(h)}$$

which gives (19). $\square$

Lemma 1 *Suppose that for all $F \in \mathcal{F}$, for all open covers $\{G_i : i \in I\}$ of F and for all $\varepsilon > 0$, there exists a finite set $\{G_{i_1}, ..., G_{i_N}\} \subset \{G_i : i \in I\}$ such that*

$$\limsup \mu_\alpha^{t_\alpha}(F) - \limsup \mu_\alpha^{t_\alpha}\Big(\bigcup_{1\leq j\leq N} \overline{G_{i_j}}\Big) < \varepsilon. \tag{21}$$

Then, $\sup_{j\in L} \overline{\Lambda}(h_j) = \overline{\Lambda}(\bigvee_{j\in L} h_j)$ for all families $\{h_j : j \in L\} \subset \mathcal{C}^T(X)$ such that $\bigvee_{j\in L} h_j \in \mathcal{C}^T(X)$.

Proof. Let $\{h_i : i \in I\} \subset \mathcal{C}^T(X)$ such that $h = \bigvee_{i\in I} h_i \in \mathcal{C}^T(X)$. If

$$\overline{\Lambda}(h) > \sup_{i\in I} \overline{\Lambda}(h_i),$$

then by Theorem 1 there exists $\lambda_0 \geq 0$, $\varepsilon_0 > 0$ and $\nu > 0$ such that

$$(\lambda_0 - \varepsilon_0) \limsup \mu_\alpha^{t_\alpha}(G^h_{\lambda_0,\varepsilon_0}) > \sup_{i\in I} \sup_{\lambda\geq 0,\varepsilon>0} \{(\lambda - \varepsilon) \limsup \mu_\alpha^{t_\alpha}(F^{h_i}_{\lambda,\varepsilon})\} + \nu$$

$$> \sup_{i\in I} \sup_{\lambda\geq 0,\varepsilon>0} \{(\lambda - \varepsilon) \limsup \mu_\alpha^{t_\alpha}(\overline{G^{h_i}_{\lambda,\varepsilon}})\} + \nu$$

$$> \sup_{i\in I} \sup_{\varepsilon>0} \{(\lambda_0 - \varepsilon + \nu/2) \limsup \mu_\alpha^{t_\alpha}(\overline{G^{h_i}_{\lambda_0,\varepsilon}})\} + \nu/2. \tag{22}$$

Take $\varepsilon_0 < \varepsilon < \varepsilon_0 + \nu/2$ in (22) and obtain

$$(\lambda_0 - \varepsilon_0) \limsup \mu_\alpha^{t_\alpha}(G^h_{\lambda_0,\varepsilon_0}) > (\lambda_0 - \varepsilon_0) \sup_{i\in I} \limsup \mu_\alpha^{t_\alpha}(\overline{G^{h_i}_{\lambda_0,\varepsilon}}) + \nu/2. \tag{23}$$

Put $F = F^h_{\lambda_0,\varepsilon_0}$, $G_i = G^{h_i}_{\lambda_0,\varepsilon}$ for all $i \in I$, and notice that $F \subset \bigcup_{i\in I} G_i$. Since $F \supset G^h_{\lambda_0,\varepsilon_0}$ we obtain by (23)

$$\limsup \mu_\alpha^{t_\alpha}(F) > \sup_{i\in I} \limsup \mu_\alpha^{t_\alpha}(\overline{G_i}) + \nu/2,$$

and so

$$\limsup \mu_\alpha^{t_\alpha}(F) > \limsup \mu_\alpha^{t_\alpha}\Big(\bigcup_{1\leq j\leq N} \overline{G_{i_j}}\Big) + \nu/2,$$

for all finite subsets $\{G_{i_j} : 1 \leq j \leq N\} \subset \{G_i : i \in I\}$, which contradicts (21). Thus,

$$\overline{\Lambda}(h) \leq \sup_{i\in I} \overline{\Lambda}(h_i).$$

$\square$

Let us now look for sufficient conditions in order that large deviation upper bounds hold with the function $l_{\mathcal{A}}$.

Proposition 3 *If for all $F \in \mathcal{F}$, for all open covers $\{G_i : i \in I\}$ of $F \cap X_0$ and for all $\varepsilon > 0$, there exists a finite set $\{G_{i_1}, ..., G_{i_N}\} \subset \{G_i : i \in I\}$ such that*

$$\limsup \mu_\alpha^{t_\alpha}(F) - \limsup \mu_\alpha^{t_\alpha}\Big(\bigcup_{1\leq j\leq N} \overline{G_{i_j}}\Big) < \varepsilon, \tag{24}$$

then the large deviation upper-bounds hold with the function $l_\mathcal{A}$, for each approximating class $\mathcal{A}$; if moreover X is completely regular, and $\mathcal{A} \subset \mathcal{C}_b(X)$, then

$$-\inf_{x\in G} l_A(x) = \inf_{0<s<\infty} \sup_{\mathcal{A}_{G,s}} \overline{\Lambda}(h) \qquad \textit{for all } G \in \mathcal{G}. \tag{25}$$

Proof. Let $\mathcal{A}$ be an approximating class, and put $f = e^{-l_\mathcal{A}}$. Suppose that

$$\delta > \nu + \sup_{x\in F} f(x)$$

for some $F \in \mathcal{F}$, and some reals δ, $\nu > 0$ with $\nu < 1$. Then, for each $x \in F \cap X_0$ there exists $h_x \in \mathcal{A} \cap \mathcal{C}^T(X)$ such that $\log(1-\nu) < h_x(x) \leq 0$ and

$$\delta > \nu + e^{\overline{\Lambda}(h_x)},$$

that is by Theorem 1,

$$\delta > \nu + \sup_{x\in F} \sup_{\lambda\geq 0, \varepsilon>0} \{(\lambda - \varepsilon) \limsup \mu_\alpha^{t_\alpha}(F_{\lambda,\varepsilon}^{h_x})\}. \tag{26}$$

By taking $\lambda = 1$ and $\varepsilon = \nu$ in (26), we obtain

$$\delta > \sup_{x\in F} \limsup \mu_\alpha^{t_\alpha}(F_{1,\nu}^{h_x}) \geq \sup_{x\in F} \limsup \mu_\alpha^{t_\alpha}(\overline{G_{1,\nu}^{h_x}}).$$

Since $F \cap X_0 \subset \bigcup_{x\in F} G_{1,\nu}^{h_x}$, the hypothesis implies

$$\delta > \limsup \mu_\alpha^{t_\alpha}(F),$$

and so

$$\limsup \mu_\alpha^{t_\alpha}(F) \leq \sup_{x\in F} f(x) \tag{27}$$

for all $F \in \mathcal{F}$, which proves the first assertion. Suppose that X is completely regular, and $\mathcal{A} \subset \mathcal{C}_b(X)$. If (25) does not hold, then

$$\sup_{x\in G} f(x) < \inf_{0<s<\infty} \sup_{\mathcal{A}_{G,s}} e^{\overline{\Lambda}(h)}$$

for some $G \in \mathcal{G}$, by Proposition 2. Since $X_0 = X$ by Proposition 1, for each $x \in G$, each real $s > 0$ and each real $t > 0$, there exists $h_{x,t} \in \mathcal{A} \cap \mathcal{C}^T(X)$ with $h_{x,t}(x) \geq -t$, and $h_s \in \mathcal{A}_{G,s}$ such that

$$\overline{\Lambda}(h_{x,t}) < \nu < \overline{\Lambda}(h_s) \tag{28}$$

for some real ν. Since $\mathcal{A} \subset \mathcal{C}_b(X)$, there exists a real $s_0 > 0$ such that $-s_0 < \inf_{y\in X} \sup_{x\in G, t>0} h_{x,t}(y)$, and $h_{s_0} \leq \bigvee_{x\in G,t>0} h_{x,t}$ (clearly, $h_{s_0}(y) \leq \sup_{x\in G,t>0} h_{x,t}(y)$ if $y \notin G$; if $h_{s_0}(y_0) > \sup_{x\in G,t>0} h_{x,t}(y_0)$ for some $y_0 \in G$, then $h_{s_0}(y_0) > \sup_{t>0} h_{y_0,t}(y_0) \geq \sup_{t>0} -t \geq 0$ which gives the contradiction). Put $k_{x,t} = h_{s_0} \wedge h_{x,t}$ for all $x \in G$ and all $t > 0$, so that $\bigvee_{x\in G,t>0} k_{x,t} = h_{s_0}$. Since $X_0 = X$, the hypothesis is equivalent to the one of Lemma 1; it follows that

$$\overline{\Lambda}(h_{s_0}) = \sup_{x\in G,t>0} \overline{\Lambda}(k_{x,t}) \leq \sup_{x\in G,t>0} \overline{\Lambda}(h_{x,t}) \leq \nu,$$

which contradicts (28). Thus, (25) holds. □

Proposition 4 *Let X be completely regular and suppose that the compact sets are Borel sets. If $\Lambda(\cdot)$ exists on some approximating class $\mathcal{A}$, then (μ_α) satisfies a vague large deviation principle with powers (t_α) and rate function $\overset{\circ}{l}_{\mathcal{A}}$.*

Proof. By Proposition 2, it suffices to show that

$$\limsup \mu_\alpha^{t_\alpha}(K) \leq \sup_K e^{-l_{\mathcal{A}}} \tag{29}$$

for all compact sets $K \subset X$. The proof of (29) is then similar to the one of the upper-bounds in Proposition 3: we use the compactness of K in place of (24). $\square$

By Proposition 2 and Proposition 3, the existence of $\Lambda(\cdot)$ on an approximating class $\mathcal{A}$ together with (24) imply a large deviation principle with rate function $\overset{\circ}{l}_{\mathcal{A}}$; the following theorem shows that $l_{\mathcal{A}}$ is in fact lower semi-continuous; moreover, these conditions are also necessary when X is completely regular.

Theorem 3 *Let $\mathcal{A}$ be an approximating class, and consider the following statements:*

(i) (μ_α) *satisfies a large deviation principle with powers* (t_α).

(ii) For all $F \in \mathcal{F}$, for all open covers $\{G_i : i \in I\}$ of $F \cap X_0$ and for all $\varepsilon > 0$, there exists a finite set $\{G_{i_1}, ..., G_{i_N}\} \subset \{G_i : i \in I\}$ such that

$$\limsup \mu_\alpha^{t_\alpha}(F) - \liminf \mu_\alpha^{t_\alpha}\Big(\bigcup_{1 \leq j \leq N} \overline{G_{i_j}}\Big) < \varepsilon.$$

(iii) $\Lambda(h)$ exists for all $h \in \mathcal{A}$, and for all $F \in \mathcal{F}$, for all open covers $\{G_i : i \in I\}$ of $F \cap X_0$ and for all $\varepsilon > 0$, there exists a finite set $\{G_{i_1}, ..., G_{i_N}\} \subset \{G_i : i \in I\}$ such that

$$\limsup \mu_\alpha^{t_\alpha}(F) - \limsup \mu_\alpha^{t_\alpha}\Big(\bigcup_{1 \leq j \leq N} \overline{G_{i_j}}\Big) < \varepsilon. \tag{30}$$

Then,

(a) (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i), and the three conditions are equivalent when X is completely regular.

(b) If (iii) holds, then (i) holds with rate function $l_{\mathcal{A}}$, and

$$-\inf_{x \in G} l_{\mathcal{A}}(x) = \inf_{0<s<\infty} \sup_{h \in \mathcal{A}_{G,s}} \Lambda(h) \qquad \textit{for all } G \in \mathcal{G}.$$

Proof. (a) Clearly, $(ii) \Rightarrow (iii)$ by Corollary 1. Suppose that (iii) holds. By Proposition 2 and Proposition 3, we have

$$\limsup \mu_\alpha^{t_\alpha}(F) \leq \sup_F e^{-l_{\mathcal{A}}} \leq \sup_G e^{-l_{\mathcal{A}}} \leq \inf_{0<s<\infty} \sup_{\mathcal{A}_{G,s}} e^{\Lambda(h)} \leq \liminf \mu_\alpha^{t_\alpha}(G) \tag{31}$$

for all $F \in \mathcal{F}$, $G \in \mathcal{G}$ with $F \subset G$, so that (i) holds with rate function $\overset{\circ}{l}_{\mathcal{A}}$. Suppose that X is completely regular. If (i) holds with some rate function J, then by Corollary 1, $\Lambda(h)$ exists for all $h \in \mathcal{A}$. For each $F \in \mathcal{F}$, each open cover $\{G_i : i \in I\}$ of F and each $\varepsilon > 0$, we have

$$\limsup \mu_\alpha^{t_\alpha}(F) \leq \sup_F e^{-J} \leq \sup_{\bigcup_{i \in I} G_i} e^{-J} = \sup_{i \in I} \sup_{G_i} e^{-J} < \sup_{i \in I} \liminf \mu_\alpha^{t_\alpha}(G_i) + \varepsilon,$$

which implies (ii), since $X_0 = X$ by Proposition 1.

(b) If (iii) holds, then we have seen above that (i) holds with rate function $\overset{\circ}{l}_{\mathcal{A}}$. Put $f = e^{-l_{\mathcal{A}}}$, and suppose that

$$f(x) + \nu < \inf_{G \supset \{x\}, G \in \mathcal{G}} \inf_{0<s<\infty} \sup_{\mathcal{A}_{G,s}} e^{\Lambda(h)}$$

for some $x \in X$ and some $0 < \nu < 1$. Then, $x \in X_0$, and there exists $h_x \in \mathcal{A} \cap \mathcal{C}^T(X)$ with $\log(1-\nu) < h_x(x) \leq 0$ such that

$$e^{\Lambda(h_x)} + \nu < \inf_{G \supset \{x\}, G \in \mathcal{G}} \inf_{0<s<\infty} \sup_{\mathcal{A}_{G,s}} e^{\Lambda(h)}. \tag{32}$$

On the other hand, by (31) we have

$$\limsup \mu_\alpha^{t_\alpha}(F^{h_x}_{\lambda,\delta}) \leq \sup_{y \in G^{h_x}_{\lambda,\varepsilon}} f(y) \leq \inf_{0<s<\infty} \sup_{\mathcal{A}_{G^{h_x}_{\lambda,\varepsilon},s}} e^{\Lambda(h)} \leq \liminf \mu_\alpha^{t_\alpha}(G^{h_x}_{\lambda,\varepsilon})$$

for all reals λ, ε, δ with $\lambda > \varepsilon > \delta > 0$, and so by Corollary 1 (with $\gamma(G^{h_x}_{\lambda,\varepsilon}) = \inf_{0<s<\infty} \sup_{\mathcal{A}_{G^{h_x}_{\lambda,\varepsilon},s}} e^{\Lambda(h)}$), we obtain

$$e^{\Lambda(h_x)} = \sup_{\lambda \geq 0, \varepsilon > 0} \{(\lambda - \varepsilon) \inf_{0<s<\infty} \sup_{\mathcal{A}_{G^{h_x}_{\lambda,\varepsilon},s}} e^{\Lambda(h)}\},$$

and by (32),

$$\sup_{\lambda \geq 0, \varepsilon > 0} \{(\lambda - \varepsilon) \inf_{0<s<\infty} \sup_{\mathcal{A}_{G^{h_x}_{\lambda,\varepsilon},s}} e^{\Lambda(h)}\} + \nu < \inf_{G \supset \{x\}, G \in \mathcal{G}} \inf_{0<s<\infty} \sup_{\mathcal{A}_{G,s}} e^{\Lambda(h)}. \tag{33}$$

Take $\lambda = 1$ and $\varepsilon = \nu$ in (33) and obtain

$$\inf_{0<s<\infty} \sup_{\mathcal{A}_{G^{h_x}_{1,\nu},s}} e^{\Lambda(h)} < \inf_{G \supset \{x\}, G \in \mathcal{G}} \inf_{0<s<\infty} \sup_{\mathcal{A}_{G,s}} e^{\Lambda(h)}$$

with $x \in G^{h_x}_{1,\nu}$, which gives the contradiction; by Proposition 2, it follows that

$$f(x) = \inf_{G \supset \{x\}, G \in \mathcal{G}} \sup_{y \in G} f(y) = \inf_{G \supset \{x\}, G \in \mathcal{G}} \inf_{0<s<\infty} \sup_{\mathcal{A}_{G,s}} e^{\Lambda(h)};$$

in particular $l_{\mathcal{A}}$ is lower semi-continuous. Now, suppose that

$$\sup_G f + \nu < \inf_{0<s<\infty} \sup_{\mathcal{A}_{G,s}} e^{\Lambda(h)} \tag{34}$$

for some $G \in \mathcal{G}$ and some $\nu > 0$. By the tightness-free version of Varadhan's theorem (see Remark 4), $e^{\Lambda(h)} = \sup_X e^h f$ for all $h \in \mathcal{C}^T(X)$. For each real $s > 0$, there exists $h \in \mathcal{A}_{G,s}$ such that

$$\sup_G f + \nu < \sup_X e^h f \leq \sup_G f \vee e^{-s} 1_{X \setminus G},$$

which gives the contradiction when $s \to +\infty$. Thus, $\sup_G f = \inf_{0<s<\infty} \sup_{\mathcal{A}_{G,s}} e^{\Lambda(h)}$ for all $G \in \mathcal{G}$, and (b) holds. $\square$

In most cases (see Examples 1, 2 and 3), the approximating class satisfies the condition of the following corollary. When $\mathcal{A} = \mathcal{C}_b(X)$, we then encounter a more familiar expression of the rate function since (36) is then equivalent to

$$J(x) = \sup_{h \in \mathcal{C}_b(X)} \{h(x) - \Lambda(h)\}. \tag{35}$$

Indeed, we have proven in [2] (Theorem 4.1) that for X normal Hausdorff, a large deviation principle is always governed by a rate function satisfying (35); this was known before under exponential tightness assumption (Remark 4) (recall that here as in [2], J is not assumed to be tight). Corollary 2 strengthens this result by relaxing the hypotheses on the topology, and by allowing $\mathcal{A}$ in (36) to be a suitable approximating class in place of $\mathcal{C}_b(X)$. It follows that for any approximating class $\mathcal{A}$ and X completely regular, the form of a rate function, which is always given by $l_{\mathcal{A}}$ (Theorem 3), generalizes (35).

Corollary 2 *Let X be completely regular. If (μ_α) satisfies a large deviation principle with powers (t_α) and rate function J, then*

$$J(x) = l_{\mathcal{A}}(x) = \sup_{h \in \mathcal{A} \cap \mathcal{C}^T(X), h(x)=0} \{-\Lambda(h)\}, \tag{36}$$

and

$$-\inf_{x \in G} J(x) = \inf_{0<s<\infty} \sup_{h \in \mathcal{A}^0_{G,s}} \Lambda(h) \quad \text{for all } G \in \mathcal{G}, \tag{37}$$

for any approximating class $\mathcal{A}$ such that for each $G \in \mathcal{G}$, each $x \in G$, and each real $s > 0$, there exists $h \in \mathcal{A}$ satisfying $e^h \leq 1_G \vee e^{-s}$ and $h(x) = 0$, where

$$\mathcal{A}^0_{G,s} = \bigcup_{x \in G} \{h \in \mathcal{A} : e^h \leq 1_G \vee e^{-s}, h(x) = 0\}. \tag{38}$$

Proof. Put $f = e^{-J}$, and define $f_0(x) = \inf_{\{h \in \mathcal{A} \cap \mathcal{C}^T(X) : h(x)=0\}} e^{\Lambda(h)}$ for all $x \in X$. By the uniqueness of the rate function on completely regular spaces, we have by Theorem 3, $f(x) = \sup_{t>0} \inf_{\{h \in \mathcal{A} \cap \mathcal{C}^T(X) : h(x) \geq -t\}} e^{\Lambda(h)}$ for all $x \in X$, with clearly, $f \leq f_0$. Suppose that $f(x) < \nu < f_0(x)$ for some $x \in X$ and some real ν. By Corollary 3.4 in [2], we have

$$e^{\Lambda(h)} = \sup_X e^h f \tag{39}$$

for all $h \in \mathcal{C}^T(X)$. Since f is upper semi-continuous, $\{f < \nu\}$ is open and so, by hypothesis, for each real $s > 0$ there exists $h_s \in \mathcal{A} \cap \mathcal{C}^T(X)$ such that $e^{h_s} \leq 1_{\{f<\nu\}} \vee e^{-s}$ and $h_s(x) = 0$. Then, $\nu < \inf_{0<s<\infty} \sup_X e^{h_s} f = \inf_{0<s<\infty} \max\{\sup_{\{f<\nu\}} e^{h_s} f, e^{-s}\}$, which gives the contradiction when $s \to +\infty$. Thus, $f = f_0$ and (36) holds. Then, (37) is a direct consequence of (39) and Theorem 3 (b). $\square$

Corollary 3 *If X is completely regular and compact, then for any approximating class $\mathcal{A}$, the following statements are equivalent:*

(i) *(μ_α) satisfies a large deviation principle with powers (t_α).*

(ii) $\limsup \mu_\alpha^{t_\alpha}(F) \leq \liminf \mu_\alpha^{t_\alpha}(G)$ *for all $F \in \mathcal{F}$, $G \in \mathcal{G}$ with $F \subset G$.*

(iii) *$\Lambda(h)$ exists for all $h \in \mathcal{A}$.*

Proof. $(i) \Rightarrow (ii)$ is clear, and $(ii) \Rightarrow (iii)$ follows from Corollary 1. If (iii) holds, then the condition (iii) of Theorem 3 holds since X is compact and $X_0 = X$, so that (i) holds. $\square$

The following definition generalizes the notion of τ-smoothness and tightness for a net of measures ([10]). The tightness of the net $(\mu_\alpha^\alpha)_{\alpha>0}$ (resp. $(\mu_n^{1/n})_{n\in\mathbb{N}^*}$) is usually called the exponential tightness of the net $(\mu_\alpha)_{\alpha>0}$ (resp. $(\mu_n)_{n\in\mathbb{N}^*}$); when there is no ambiguity about the net of powers, "exponential tightness (resp. smoothness)" will mean "tightness (resp. smoothness) of $(\mu_\alpha^{t_\alpha})$".

Definition 2

- The net $(\mu_\alpha^{t_\alpha})$ is *τ-smooth* if for all open covers $\{G_i : i \in I\}$ of X and for all $\varepsilon > 0$, there exists a finite set $\{G_{i_1}, ..., G_{i_N}\} \subset \{G_i : i \in I\}$ such that
$$\limsup \mu_\alpha^{t_\alpha}(X \backslash \bigcup_{1\le j\le N} G_{i_j}) < \varepsilon.$$
- The net $(\mu_\alpha^{t_\alpha})$ is *tight* if for all $\varepsilon > 0$ there exists a compact Borel set $K \subset X$ such that
$$\limsup \mu_\alpha^{t_\alpha}(X \backslash K) < \varepsilon.$$

For X completely regular, the condition (iii) in the following corollary (which is exactly the set-theoretical condition in (iii) of Theorem 3 with $X = X_0$) is in general strictly weaker than the tightness of $(\mu_\alpha^{t_\alpha})$ since it is also necessary for large deviations. In particular, (b) improves Bryc's theorem (see Remark 4).

Corollary 4 *Consider the following statements:*

(i) $(\mu_\alpha^{t_\alpha})$ *is tight.*

(ii) $(\mu_\alpha^{t_\alpha})$ *is τ-smooth.*

(iii) For all $F \in \mathcal{F}$, for all open covers $\{G_i : i \in I\}$ of F and for all $\varepsilon > 0$, there exists a finite set $\{G_{i_1}, ..., G_{i_N}\} \subset \{G_i : i \in I\}$ such that
$$\limsup \mu_\alpha^{t_\alpha}(F) - \limsup \mu_\alpha^{t_\alpha}(\bigcup_{1\le j\le N} \overline{G_{i_j}}) < \varepsilon. \tag{40}$$

Then,

(a) $(i) \Rightarrow (ii) \Rightarrow (iii)$. *If X is locally compact and either Hausdorff or regular, then* $(i) \Leftrightarrow (ii)$.

(b) If X is completely regular and $\Lambda(\cdot)$ exists on some approximating class $\mathcal{A}$, then any above condition implies a large deviation principle with powers (t_α) and rate function
$$l_{\mathcal{A}}(x) = \inf_{t>0} \sup_{\{h\in\mathcal{A}\cap\mathcal{C}^T(X):h(x)\ge -t\}} \{-\Lambda(h)\} \qquad \text{for all } x \in X. \tag{41}$$

Proof. $(i) \Rightarrow (ii)$ is clear. Suppose that (ii) holds. Let $F \in \mathcal{F}$, $\{G_i : i \in I\}$ be an open cover of F, and $\varepsilon > 0$. Then, $\bigcup_{i\in I} G_i \cup X\backslash F$ is an open cover of X, and so there is a finite subset $\{G_{i_1}, ..., G_{i_N}\} \subset \{G_i : i \in I\}$ such that
$$\limsup \mu_\alpha^{t_\alpha}(X \backslash (\bigcup_{1\le j\le N} G_{i_j} \cup X\backslash F)) = \limsup \mu_\alpha^{t_\alpha}(F \backslash \bigcup_{1\le j\le N} G_{i_j}) < \varepsilon,$$

which implies (40), and (iii) holds; when X is locally compact and either Hausdorff or regular, there exists a basis of relatively compact open sets ([5]), and the τ-smoothness property with this basis gives (i); this proves (a). Since $X = X_0$ for X completely regular, (b) follows from (a) and Theorem 3. $\square$

Example 3 Let X be completely regular. Suppose that $(\mu_\alpha^{t_\alpha})$ satisfies the condition (iii) of Corollary 4, and $\Lambda(n\cdot)$ exists on a set of Urysohn functions $\mathcal{A}' = \{h_{G,x} : G \in \mathcal{G}, x \in G, 1_{\{x\}} \le h_{G,x} \le 1_G\}$ for each $n \in \mathbb{N}^*$. Then, $\Lambda(\cdot)$ exists on the approximating class $\mathcal{A} = \{nh-n; h \in \mathcal{A}', n \in \mathbb{N}^*\}$, which moreover satisfies the hypothesis of Corollary 2, so that a large deviation principle holds with rate function

$$J(x) = \sup_{\{h\in\mathcal{A}', n\in\mathbb{N}^*; h(x)=1\}} \{n - \Lambda(nh)\}.$$

Note that $\mathcal{A}$ being not necessarily stable by finite minima, even in presence of Hausdorffness and exponential tightness, the strong form of Bryc's theorem with a well-separating class in place of $\mathcal{C}_b(X)$ (see Remark 4) does not apply; the finite minima property is however a crucial argument in it's proof.

The following corollary is a Prohorov-type result where the tightness of $(\mu_\alpha^{t_\alpha})$ is replaced by the τ-smoothness. The net version is the analogue for large deviations of a Topsøe's theorem which states that an eventually bounded net of positive Borel measures on a regular Hausdorff space is narrowly compact if it is τ-smooth ([10], Theorem 9.2); a somewhat stronger result for X normal Hausdorff is given in [2] (see Remark 3).

Suppose now that the directed set defining the net $(\mu_\alpha^{t_\alpha})$ has a cofinal sequence; if X is regular normal second countable and $(\mu_\alpha^{t_\alpha})$ is τ-smooth, it follows from Example 1 that a large deviation principle holds for some subsequence $(\mu_{\alpha_n}^{t_{\alpha_n}})$; when X is regular Hausdorff second countable and $(\mu_\alpha^{t_\alpha})$ is tight, this is well known ([3], Lemma 4.1.23; [6], [8] and [4] for $(\mu_\alpha^{t_\alpha})$ a sequence and X Polish); whatever the proofs, these results are all obtained by using in a crucial way the exponential tightness assumption, so that the Hausdorff hypothesis cannot be dropped. Since there exist non Hausdorff spaces satisfying conditions of Example 1 ([9]), it follows that under normality assumption, the sequential version of Corollary 5 improves them by weakening the tightness hypothesis and allowing no Hausdorff spaces (see Remark 3 (b)).

Corollary 5 *Let X be completely regular. If $(\mu_\alpha^{t_\alpha})_{\alpha\in\wp}$ is τ-smooth, then (μ_α) has a subnet (μ_β) satisfying a large deviation principle with powers (t_β). If moreover X has an approximating class which is separable with respect to the uniform metric, and $\wp$ has a cofinal sequence, then (μ_β) can be chosen as a sequence.*

Proof. Let $\mathcal{A}$ be an approximating class, and define $\Lambda_\alpha(h) = \log \mu_\alpha^{t_\alpha}(e^{h/t_\alpha})$ for all $h \in \mathcal{A}$. Then, $(\Lambda_\alpha(\cdot))$ is a net in the compact space $[-\infty, +\infty]^{\mathcal{A}}$ (with the product topology), and so there is a subnet $(\Lambda_\beta(\cdot))$ converging to some limit $\Lambda'(\cdot)$ on $\mathcal{A}$. The first assertion follows from Corollary 4 applied to $(\mu_\beta^{t_\beta})$. Suppose that $\mathcal{A}$ contains a countable uniformly dense set $\mathcal{D}$, and $\wp$ has a cofinal sequence (α_m). Since $[-\infty, +\infty]^{\mathcal{D}}$ is sequentially compact, $(\Lambda_{\alpha_m}(\cdot))$ has a subsequence $(\Lambda_\beta(\cdot))$ converging to some limit $\Lambda'(\cdot)$ on $\mathcal{D}$. Let $h \in \mathcal{A}$ and (h_n) be a sequence in $\mathcal{D}$ such that $\lim ||h - h_n|| = 0$. For each $n \in \mathbb{N}$, we have

$$|\log \mu_\beta^{t_\beta}(e^{h/t_\beta}) - \log \mu_\beta^{t_\beta}(e^{h_n/t_\beta})| \le ||h - h_n||$$

and

$$\Lambda'(h_n) - ||h - h_n|| \leq \log \liminf \mu_\beta^{t_\beta}(e^{h/t_\beta})$$
$$\leq \log \limsup \mu_\beta^{t_\beta}(e^{h/t_\beta}) \leq \Lambda'(h_n) + ||h - h_n||,$$

which implies that $\lim \Lambda'_\beta(h) = \Lambda'(h)$ exists with $\Lambda'(h) = \lim \Lambda'(h_n)$. The last assertion follows by applying Corollary 4 to $(\mu_\beta^{t_\beta})$. □

Remark 2 The condition *(iii)* in Corollary 4 has to be compared with a similar one given in Theorem 4.1 of [2]; it is shown there that if $\Lambda(\cdot)$ exists on $\mathcal{C}_b(X)$ with X normal Hausdorff, then the following condition is equivalent to large deviations:

(iii') For all $F \in \mathcal{F}$, for all open covers $\{G_i : i \in I\}$ of F and for all $\varepsilon > 0$, there exists a finite set $\{G_{i_1}, ..., G_{i_N}\} \subset \{G_i : i \in I\}$ such that

$$\liminf \mu_\alpha^{t_\alpha}(\overset{\circ}{F}) - \limsup \mu_\alpha^{t_\alpha}(\bigcup_{1 \leq j \leq N} \overline{G_{i_j}}) < \varepsilon. \tag{42}$$

Of course, (iii') is weaker than (iii), and this is due to the assumptions of normality, Hausdorffness and existence of $\Lambda(\cdot)$ on all continuous bounded functions, and in particular on those considered in the following interpolation property characterizing normal spaces: if f and g are real-valued respectively upper and lower semi-continuous functions on X such that $f \leq g$, then there is a continuous function h satisfying $f \leq h \leq g$.

Remark 3

(a) In the same way that $(ii) \Rightarrow (iii)$ in Corollary 4, it is easy to see that the condition (iii') in Remark 2 is implied by the following one, which is weaker than the τ-smoothness:

(ii') For all open covers $\{G_i : i \in I\}$ of X and for all $\varepsilon > 0$, there exists a finite set $\{G_{i_1}, ..., G_{i_N}\} \subset \{G_i : i \in I\}$ such that

$$\limsup \mu_\alpha^{t_\alpha}(X \backslash \bigcup_{1 \leq j \leq N} \overline{G_{i_j}}) < \varepsilon.$$

When X is normal Hausdorff, the first assertion of Corollary 5 holds verbatim by replacing the τ-smoothness by (ii') ([2], Corollary 4.4). The proof is analogue and based on Theorem 4.1 of [2] (see Remark 2) together with $(ii') \Rightarrow (iii')$. In particular, the existence of $\Lambda(\cdot)$ is required on all $\mathcal{C}_b(X)$, and this does not allow to get sequential versions; reducing the domain of existence of $\Lambda(\cdot)$ from $\mathcal{C}_b(X)$ to approximating classes makes the job.

(b) The case where $(\mu_\alpha^{t_\alpha})$ is a sequence and X Polish or Hausdorff second countable is studied in detail in [7] where large deviations for some subsequence are obtained with a weaker condition than tightness ([7], Theorem 3.1); however, the proof requires compact Borel sets.

Remark 4 Bryc's theorem states that for X completely regular Hausdorff, the existence of $\Lambda(\cdot)$ on all $\mathcal{C}_b(X)$ together with the exponential tightness imply a large deviation principle with (necessarily tight) rate function J satisfying (35); moreover, $\Lambda(h) = \sup_{x \in X}\{h(x) - J(x)\}$ for all $h \in \mathcal{C}_b(X)$ ([1], [3] Theorem 4.4.2).

(a) Clearly, Corollary 2 and Corollary 4 together strengthen the first assertion in three ways (besides allowing general nets): by removing the Hausdorff hypothesis, by weakening the exponential tightness one, and by replacing $\mathcal{C}_b(X)$ by any approximating class (and (35) by (41)). The last assertion follows from the large deviation property, and in particular it holds without any tightness or topological hypothesis, as states the following generalization of Varadhan's theorem ([2], Corollary 3.4): for any topological space X and any function $l : X \to [0, +\infty]$ satisfying

$$\limsup \mu_\alpha^{t_\alpha}(F) \leq \sup_{x \in F} e^{-l}(x) \leq \liminf \mu_\alpha^{t_\alpha}(G)$$

$$(\text{resp.} \quad \limsup \mu_\alpha^{t_\alpha}(F) \leq \sup_{x \in G} e^{-l}(x) \leq \liminf \mu_\alpha^{t_\alpha}(G))$$

for all $F \in \mathcal{F}, G \in \mathcal{G}$ with $F \subset G$, we have $\Lambda(h) = \sup_{x \in X}\{h(x) - l(x)\}$ for all $h \in \mathcal{C}^T(X)$.

When X is normal Hausdorff and $\Lambda(\cdot)$ exists on all $\mathcal{C}_b(X)$, Bryc's theorem has been improved in [2] in a more satisfactory way since (ii') (resp. (iii')) implies (resp. is equivalent to) large deviations, and is weaker than (ii) (resp. (iii)) (see Remark 2 and Remark 3).

(b) The conclusions of Bryc's theorem hold replacing in the hypothesis $\mathcal{C}_b(X)$ by a well-separating class (i.e. a class of real-valued continuous functions, containing the constants, stable by finite minima, and separating the points) ([3], Theorem 4.4.10); our results do not concern this variant, which is strongly based on the exponential tightness assumption.

References

[1] W. Bryc. Large deviations by the asymptotic value methods, *Diffusion Processes and Related Problems in Analysis, 447-472. Birkhäuser, Basel, Switzerland, 1990.*

[2] H. Comman. Criteria for large deviations. *Trans. Amer. Math. Soc. 355 (2003), 2905-2923.*

[3] A. Dembo and O. Zeitouni. Large deviations techniques and applications, *Second edition, Springer-Verlag, 1998.*

[4] P. Dupuis and R. S. Ellis. A weak convergence approach to the theory of large deviations, *Wiley, 1997.*

[5] J. L. Kelley. General topology, *Springer-Verlag, 1955.*

[6] G. L. O'Brien G. L. and W. Vervaat. Capacities, large deviations and loglog laws, *Stable Processes and Related Topics, Birkhauser, Boston (1991) 43-84.*

[7] G. L. O'Brien. Sequence of capacities with connections to large deviation theory, *Journal of Theoretical Probability, 9 (1996) 19-35.*

[8] A. A. Pukhalskii. On functional principle of large deviations, *New Trends in Probability and Statistics, V. Sazonov and T. Shervashidze, Editors, 198-218, VSP-Mokslas, Zeist, The Netherlands, 1991.*

[9] L. A. Steen and J. Arthur Seebach, Jr. Counterexamples in topology, *Holt, Rinehart and Winston, Inc. 1970.*

[10] F. Topsøe. Topology and measure, *Springer-Verlag, 1970.*