Capacities on C*-algebras

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Abstract

We distinguish three classes of capacities on a C*-algebra: subadditive, additive and maxitive. A tightness notion for capacities, the vague and narrow topologies on the set of capacities are introduced. The vague space of additive capacities which are finite on compact projections is a non commutative version of the usual vague space of Radon measures on a locally compact Hausdorff space X. We give criterions of vague and narrow relative compactness in various classes of capacities. This allows us to extend most classical compactness theorems for Radon measures. The set of bounded (resp. tight) maxitive capacities is in bijection with the set of positive q-upper semicontinuous (resp. strongly q-upper semicontinuous) operators. This allows us to define a vague (resp. narrow) large deviation principle for a net of capacities as a vague (resp. narrow) convergence of this net towards a maxitive capacity, generalizing the classical notion for Radon probability measures on X. Next, we apply compactness theorems in order to extend some results in large deviations theory.

Introduction

Recently, various authors ([11], [12], [13], [14]) have developed a new topological theory of set-capacities on a locally compact Hausdorff space X, which includes important results of the topological theory of Radon measures as well as the one of large deviations.

The aim of this work is to develop a similar theory in the general setting of non commutative C*-algebras, in view to obtain a generalization of the above mentioned results about measures and large deviations.

Let us recall briefly how works this theory. Let $\mathcal{K}$ (resp. $\mathcal{G}$, $\mathcal{F}$) denote the set of compact (resp. open, closed) subsets of X. In the sense of [12], a set-capacity on X is a map γ from the power set of X to $[0, \infty]$ such that:

(i) $\gamma(\emptyset) = 0$,

(ii) $\gamma(Y) = \sup_{K \subset Y, K \in \mathcal{K}} \gamma(K)$ for all $Y \subset X$ (inner regularity),

(iii) $\gamma(K) = \inf_{G \supset K, G \in \mathcal{G}} \gamma(G)$ for all $K \in \mathcal{K}$ (outer regularity on $\mathcal{K}$).

The vague (resp. narrow) topology on the set of set-capacities is the coarsest topology for which the mappings $\gamma \mapsto \gamma(Y)$ are upper semi continuous (usc) for all $Y \in \mathcal{K}$ (resp. $\mathcal{F}$) and lower semi continuous (lsc) for all $Y \in \mathcal{G}$.

The basic scheme is to establish general compactness theorems, and next to identify some classes of set-capacities as classes of measures, or relevant objects connected

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with large deviations theory (e.g. rate functions, powers of measures). Then, most of classical (vague and narrow) compactness theorems for these measures as well as large deviations results are easily derived.

For example, the vague space of $[0,1]$-valued set-capacities is compact Hausdorff, and a tight (i.e. for all $\varepsilon > 0$, eventually $\gamma_\alpha(X \backslash K) < \varepsilon$ for some compact $K \subset X$) vaguely convergent net (γ_α) of subadditive capacities (i.e. $\gamma(K_1 \cup K_2) \leq \gamma(K_1) + \gamma(K_2)$ for all K_1, K_2 in $\mathcal{K}$) is narrowly convergent. These results can be applied in the following way:

1. (Measures case) The narrow space of additive capacities γ (i.e. γ subadditive with $\gamma(K_1 \cup K_2) = \gamma(K_1) + \gamma(K_2)$ if $K_1 \cap K_2 = \emptyset$) such that $\gamma(X) = 1$, is homeomorphic to the usual narrow space of Radon probability measures on X, and closed in the narrow space of set-capacities. Then, we obtain the direct Prohorov theorem: a tight set of Radon probability measures is narrowly relatively compact.
2. (Large deviations case) A family of Radon probability measures $\{\mu_\alpha; \alpha > 0\}$ on X satisfies a vague (resp. narrow) large deviation principle with powers $\{t_\alpha; \alpha > 0\} \subset]0, \infty[$ (where $\lim_{\alpha \to 0} t_\alpha) = 0$) if there exists a positive usc function f on X such that:
   (i) $\limsup \mu_\alpha(Y)^{t_\alpha} \leq \sup_{t \in Y} f(t)$ for all $Y \in \mathcal{K}$ (resp. $\mathcal{F}$),
   (ii) $\liminf \mu_\alpha(G)^{t_\alpha} \geq \sup_{t \in G} f(t)$ for all $G \in \mathcal{G}$.

   There is a bijection between the set of bounded maxitive set-capacities γ (i.e. $\gamma(\bigcup_{i \in I} G_i) = \sup_{i \in I} \gamma(G_i)$ for all $\{G_i; i \in I\} \subset \mathcal{G}$) and the set of positive bounded usc functions f given by the relation $\gamma(Y) = \sup_{t \in Y} f(t)$ for all $Y \in \mathcal{F} \cup \mathcal{G}$. In this correspondence, a bounded maxitive set-capacity is tight if and only if its associated usc function f satisfies: $\{t; f(t) \geq s\}$ is compact for all $s > 0$. For all $\alpha > 0$, the set-function $\mu_\alpha^{t_\alpha}$ defined by $\mu_\alpha^{t_\alpha}(Y) = \mu_{\alpha,*}(Y)^{t_\alpha}$ where $\mu_{\alpha,*}(Y) = \sup_{K \subset Y, K \in \mathcal{K}} \mu_\alpha(K)$ for all $Y \subset X$, is a subadditive set-capacity. Thus, the net (μ_α) satisfies a vague (resp. narrow) large deviation principle with powers (t_α) if and only if the net of set-capacities $(\mu_\alpha^{t_\alpha})$ converges vaguely (resp. narrowly) to a maxitive set-capacity. Now, if (μ_α) satisfies a vague large deviation principle, and if $(\mu_\alpha^{t_\alpha})$ is tight, then the principle is narrow, the maxitive set-capacity limit is tight, and so the associated governing function f satisfies: $\{t; f(t) \geq s\}$ is compact for all $s > 0$. This is a well known result in large deviation theory where the tightness of $(\mu_\alpha^{t_\alpha})$ is known as the exponential tightness of the family $\{\mu_\alpha; \alpha > 0\}$.

Notice that the whole theory works similarly if we restrict the definition of a set-capacity to $\mathcal{G} \cup \mathcal{F}$. However, if we extend directly this restricted definition to the non commutative context of C*-algebras using open, closed and compact projections, then in general positive functionals will not be capacities by the lack of inner regularity on closed projections.

Now, change the definition of a set-capacity allowing outer regularity (in place of inner regularity) on closed sets. More precisely, in our sense a set-capacity on X is a map γ from $\mathcal{G} \cup \mathcal{F}$ to $[0, \infty]$ such that:

(i) $\gamma(\emptyset) = 0$,

(ii) $\gamma(G) = \sup_{K \subset G, K \in \mathcal{K}} \gamma(K)$ for all $G \in \mathcal{G}$ (inner regularity on $\mathcal{G}$),

(iii) $\gamma(F) = \inf_{G \supset F, G \in \mathcal{G}} \gamma(G)$ for all $F \in \mathcal{F}$ (outer regularity on $\mathcal{F}$).

Then, since both definitions coincide on compact and open sets, the vague theory is the same. For the narrow case, we obtain some simplifications and improvements.

For example, the narrow space of these set-capacities is regular when X is normal; this gives an equivalence between the notions of relative compactness and (in general weaker) net-compactness. We prove a very simple criterion of narrow net-compactness in the set of bounded capacities (Theorem 4) where tightness is replaced by a certain notion of uniform inner regularity. Finally, we obtain the same results about measures and large deviations.

It turns out that a large part of this new theory can be extended to C*-algebras, using the non commutative topology. In this paper, we expound it for a general C*-algebra $\mathcal{U}$, and of course the above case is recovered taking $\mathcal{U} = \mathcal{C}_0(X)$ the algebra of complex continuous functions on X vanishing at infinity.

1 Capacities spaces

Let $\mathcal{U}$ be a C*-algebra, $\tilde{\mathcal{U}}$ the algebra obtained by adjoining a unit 1 to $\mathcal{U}$, and $\mathcal{U}^{**}$ the universal enveloping von Neumann algebra of $\mathcal{U}$. For $M \subset \mathcal{U}^{**}$, M_h denotes the selfadjoint part of M, and M_+ the positive part of M. A projection $b \in \mathcal{U}^{**}$ is *open* if there is an increasing net in $\mathcal{U}_+$ converging strongly to b. A projection $c \in \mathcal{U}^{**}$ is *closed* if $1 - c$ is open. A closed projection $a \in \mathcal{U}^{**}$ is *compact* if there exists $x \in \mathcal{U}_+$ such that $a \leq x$ ([4],[15]). It is well known that the supremum of a family of open projections is an open projection, and the supremum of two commuting closed projections is a closed projection. Let A (resp. B, C) denote the set of compact (resp. open, closed) projections in $\mathcal{U}^{**}$.

Definition 1 A map $\gamma : B \cup C \to [0, \infty]$ is a *capacity* on $\mathcal{U}$ if

(i) $\gamma(0) = 0$,

(ii) $\gamma(b) = \sup_{a \leq b, a \in A} \gamma(a)$ for all $b \in B$ (inner regularity on B),

(iii) $\gamma(c) = \inf_{b \geq c, b \in B} \gamma(b)$ for all $c \in C$ (outer regularity on C).

Denote by Γ the set of capacities on $\mathcal{U}$. We say that $\gamma \in \Gamma$ is

- *subadditive* if

$$\sup_{a \leq a_1 \vee a_2, a \in A} \gamma(a) \leq \gamma(a_1) + \gamma(a_2) \qquad \text{for all } a_1, a_2 \text{ in } A \tag{1}$$

- *additive* if γ is subadditive and

$$\gamma(a_1 \vee a_2) = \gamma(a_1) + \gamma(a_2) \qquad \text{for all } a_1, a_2 \text{ in } A \text{ with } a_1 a_2 = 0. \tag{2}$$

- *maxitive* if

$$\gamma(\bigvee_{i \in I} b_i) = \bigvee_{i \in I} \gamma(b_i) \qquad \text{for all families } \{b_i ; i \in I\} \subset B. \tag{3}$$

We denote by Γ_{sa} (resp. Γ_a, Γ_m) the set of subadditive (resp. additive, maxitive) capacities on $\mathcal{U}$. For $\Gamma' \subset \Gamma$ and $\Delta \subset [0, +\infty]$, Γ'_Δ denotes the set of Δ-valued capacities in Γ', and $\Gamma'_<$ (resp. Γ'_1) the set of capacities in Γ' such that $\gamma(a) < \infty$ for all $a \in A$ (resp. $\gamma(1) = 1$).

Recall that a positive Choquet's set-capacity on X is a map γ from the powerset of X to $[0,\infty]$ such that $\gamma(\bigcup_n Y_n) = \sup_n \gamma(Y_n)$ for each increasing sequence (Y_n) of subsets of X, and $\gamma(\bigcap_n K_n) = \inf_n \gamma(K_n)$ for each decreasing sequence (K_n) in $\mathcal{K}$ ([7]). When γ satisfy (1) and (2), γ is said additive. When

$$\gamma(Y) = \inf_{G \supset Y, G \in \mathcal{G}} \gamma(G) \qquad \text{for all } Y \subset X,$$

γ is said outer regular. The following proposition shows that additive capacities are non-commutative versions of positive, additive and outer regular Choquet's set-capacities vanishing at $\emptyset$.

Proposition 1 *Let $\mathcal{U} = \mathcal{C}_0(X)$. For each $\gamma \in \Gamma_a$, let γ^* defined by $\gamma^*(Y) = \inf_{G \supset Y, G \in \mathcal{G}} \gamma(G)$ for all $Y \subset X$. Then, the map $\gamma \mapsto \gamma^*$ is a bijection between Γ_a and the set of additive and outer regular Choquet's set-capacities vanishing at $\emptyset$; the inverse map is the restriction to $\mathcal{F} \cup \mathcal{G}$.*

Proof. Let $\gamma \in \Gamma_a$. Then, $\gamma_{|\mathcal{K}}$ is a positive, increasing, additive, and right continuous (i.e. for all $K \in \mathcal{K}$, for all t with $\gamma_{|\mathcal{K}}(K) < t$, there exists $G \in \mathcal{G}$, $G \supset K$ such that $\gamma_{|\mathcal{K}}(K') < t$ for all compact $K' \subset G$) map defined on $\mathcal{K}$. By Theorem 42 of [7], the set function ${\gamma_{|\mathcal{K}}}^*$ defined by:

(i) ${\gamma_{|\mathcal{K}}}^*(G) = \sup_{K \subset G, K \in \mathcal{K}} \gamma_{|\mathcal{K}}(K)$ for all $G \in \mathcal{G}$,

(ii) ${\gamma_{|\mathcal{K}}}^*(Y) = \inf_{G \supset Y, G \in \mathcal{G}} {\gamma_{|\mathcal{K}}}^*(G)$ for all $Y \subset X$,

is a Choquet's set-capacity. Since ${\gamma_{|\mathcal{K}}}^*$ coincides with γ on $\mathcal{F} \cup \mathcal{G}$, we have ${\gamma_{|\mathcal{K}}}^* = \gamma^*$. It follows that $\gamma \mapsto \gamma^*$ is an injective map from Γ_a to the set of additive and outer regular Choquet's set-capacities vanishing at $\emptyset$; since each such a set-capacity restricted to $\mathcal{F} \cup \mathcal{G}$ is in Γ_a by Choquet's capacitability theorem, the proposition is proved. $\square$

Notice that each positive functional ω on $\mathcal{U}$ is a capacity. Indeed, it is proved in [1] that ω is inner regular on B (and so outer regular on C) when $1 \in \mathcal{U}$. Suppose $1 \notin \mathcal{U}$, and let p be a projection in $\mathcal{U}^{**}$. It is known that p is open if and only if p is open in $\tilde{\mathcal{U}}^{**} (\simeq \mathcal{U}^{**} \oplus \mathbf{C})$, and that p is compact if and only if p is closed in $\tilde{\mathcal{U}}^{**}$ ([4]). Then, working in $\tilde{\mathcal{U}}^{**}$ with the canonical extension $\tilde{\omega}$ makes the job.

Definition 2 The *vague* (resp. *narrow*) topology on Γ is the coarsest topology for which the mappings $\gamma \mapsto \gamma(p)$ are upper semi-continuous for all $p \in A$ (resp. C), and lower semi-continuous for all $p \in B$.

Recall that a Radon measure on X is a Borel measure μ satisfying $\mu(Y) = \sup_{K \subset Y, K \in \mathcal{K}} \mu(K)$ for all Borel subsets $Y \subset X$, and $\mu(K) < \infty$ for all $K \in \mathcal{K}$. This definition is taken from [3], and since X is locally compact, it coincides with Bourbaki's definition ([3], Notes and Remarks pp. 61). Notice that a (unbounded) Radon measure μ on X is not necessarily a set-capacity by the lack of outer regularity on $\mathcal{F}$. However, Theorem 1 shows that the vague space of additive capacities which are finite on compact projections is a non commutative version of the usual vague space of Radon measures on X.

Theorem 1 *Let $\mathcal{U} = \mathcal{C}_0(X)$. Then, $\Gamma_{a,<}$ provided with the vague topology is homeomorphic to the vague space of Radon measures on X. The set $\Gamma_{a,[0,\infty[}$ provided with the narrow topology is homeomorphic to the narrow space of bounded Radon measures on X.*

Proof. It is well known ([16]) that the set of Radon measures on X is in bijection with the set $\mathcal{M}$ of Borel measures μ on X satisfying

(i) $\mu(K) < \infty$ for all $K \in \mathcal{K}$,

(ii) $\mu(G) = \sup_{K\subset G, K\in\mathcal{K}} \mu(K)$ for all $G \in \mathcal{G}$,

(iii) $\mu(Y) = \inf_{G\supset Y, G\in\mathcal{G}} \mu(G)$ for all Borel subsets $Y \subset X$.

More precisely, if μ is a Radon measure, then $\mu^* \in \mathcal{M}$ where $\mu^*(Y) = \inf_{G\supset Y, G\in\mathcal{G}} \mu(G)$ for all Borel subsets $Y \subset X$. Conversely, if $\mu \in \mathcal{M}$, then μ_* is a Radon measure where $\mu_*(Y) = \sup_{K\subset Y, K\in\mathcal{K}} \mu(K)$ for all Borel subsets $Y \subset X$. Since in this correspondence the values on $\mathcal{K} \cup \mathcal{G}$ are preserved, it suffices to show that $\Gamma_{a,<}$ is in bijection with $\mathcal{M}$. Let $\gamma \in \Gamma_{a,<}$. We will show that for any pair (K_0, K_1) in $\mathcal{K}$ with $K_0 \subset K_1$, we have

$$\gamma(K_1) - \gamma(K_0) = \sup_{K\subset K_1\setminus K_0, K\in\mathcal{K}} \gamma(K).$$

Let K_0, K_1 in $\mathcal{K}$ with $K_0 \subset K_1$, and suppose

$$\gamma(K_0) + \sup_{K\subset K_1\setminus K_0, K\in\mathcal{K}} \gamma(K) > \gamma(K_1).$$

Then, there is $K \in \mathcal{K}$ with $K \subset K_1\setminus K_0$ such that

$$\gamma(K_0) + \gamma(K) > \gamma(K_1) \geq \gamma(K_0 \cup K)$$

which contradicts additivity. If there exists $r > 0$ such that

$$\gamma(K_0) + \sup_{K\subset K_1\setminus K_0, K\in\mathcal{K}} \gamma(K) < r < \gamma(K_1),$$

then by outer regularity there exists an open set $U \supset K_0$ such that

$$\gamma(U) + \sup_{K\subset K_1\setminus K_0, K\in\mathcal{K}} \gamma(K) < r < \gamma(K_1).$$

Put $V = U \cap K_1$. We have $K_1 = V \cup (K_1\setminus K_0)$, and by Wilker property in the Hausdorff space K_1, there are K_2, K_3 in $\mathcal{K}$ such that $K_2 \subset V$, $K_3 \subset (K_1\setminus K_0)$ and $K_1 = K_2 \cup K_3$. We obtain

$$\gamma(K_1) \leq \gamma(K_2) + \gamma(K_3) \leq \gamma(U) + \sup_{K\subset K_1\setminus K_0, K\in\mathcal{K}} \gamma(K) < r < \gamma(K_1)$$

and the contradiction. We have shown that $\gamma_{|\mathcal{K}}$ is a Radon content and by Theorem 1.4 of [3], μ_γ defined by $\mu_\gamma(Y) = \sup_{K\subset Y, K\in\mathcal{K}} \gamma(K)$ for all Borel subsets $Y \subset X$ is a Radon measure on X. Thus, $\mu^*_\gamma \in \mathcal{M}$ and coincides with γ on $\mathcal{G} \cup \mathcal{F}$. Conversely, if $\mu \in \mathcal{M}$, then $\gamma = \mu_{|\mathcal{G}\cup\mathcal{F}} \in \Gamma_{a,<}$ and $\mu^*_\gamma = \mu$. The second assertion is clear since a bounded Radon measure is regular. □

Let $x \in \mathcal{U}_h^{**}$ and E_Y^x denote the spectral projection of x corresponding to the borel set $Y \subset \mathbb{R}$. Following [4], we say that x is *q-upper semicontinuous* (q-usc) if $E^x_{]-\infty,t[} \in B$ for all $t \in \mathbb{R}$; if moreover $E^x_{[t,\infty[} \in A$ for all $t > 0$, x is *strongly* q-usc. We say that x is *q-lower semicontinuous* (q-lsc) (resp. *strongly* q-lsc) if $-x$ is q-usc (resp. strongly q-usc). In [5] we proved the following Theorem 2 which establish a bijection between the set of positive q-usc operators and $\Gamma_{m,[0,\infty[}$, as well as the general form of a bounded maxitive capacity. Suppose that $\mathcal{U} = \mathcal{C}_0(X)$, and let z be a positive q-usc operator corresponding to some positive bounded usc function f on X, $p \in B \cup C$ corresponding to some $Y \in \mathcal{G} \cup \mathcal{F}$, and γ the maxitive capacity represented by z as in Theorem 2. Then, $\gamma(p) = \sup_{t \in Y} f(t)$ ([5], Proposition 1), and therefore (4) is the general noncommutative version of this expression.

Theorem 2 *Let γ be a map from $B \cup C$ to $[0, \infty]$. Then the following properties are equivalent:*

(i) γ is a bounded maxitive capacity.

(ii) There exists a positive q-usc operator z such that

$$\forall p \in B \cup C, \qquad \gamma(p) = \sup\{\lambda \in \sigma(z); \forall \varepsilon > 0, pE^z_{]\lambda-\varepsilon,\lambda+\varepsilon]} \neq 0\} \tag{4}$$

We say that z represents γ. Furthermore, there is a unique positive q-usc operator which represents γ.

Corollary 1 *Theorem 2 establishes a bijection between the set of closed projections and the set of $\{0,1\}$-valued maxitive capacities. More precisely, any such a capacity γ is represented by a unique closed projection c in the following way:*

$$\forall p \in B \cup C, \qquad \gamma(p) = 0 \Leftrightarrow pc = 0. \tag{5}$$

Proof. Clearly, a closed projection defines by (4) a $\{0,1\}$-valued maxitive capacity. Reciprocally, let $\gamma \in \Gamma_{m,\{0,1\}}$. If for all $b \in B$, $b \neq 0$, $\gamma(b) = 1$, clearly the identity 1 represents γ. If $\gamma = 0$, 0 represents γ. Then, $E'_0 = \bigvee\{b \in B; \gamma(b) \leq 0\}$ is an open projection such that $\gamma(E'_0) = 0$ by maxitivity. Put $F = 1 - E'_0$, we claim that F is the q-usc operator which represents γ. Indeed, let $p \in B \cup C$; if $\gamma(p) = 0$, then $p \leq E'_0$ (this is clear if $p \in B$, and if $p \in C$, by outer regularity there exists $b \in B$, $b \geq c$ with $\gamma(b) = 0$) and so $p(1 - E^F_{1-\varepsilon}) = 0$ for all $0 < \varepsilon \leq 1$, and since $p \leq E^F_\varepsilon$ for all $\varepsilon > 0$, we obtain (4) for $\gamma(p)$; if $\gamma(p) = 1$, then $p \not\leq E'_0$ and so $p(1 - E^F_{1-\varepsilon}) \neq 0$ for all $\varepsilon > 0$, which again gives (4). □

The following definition generalizes the notion of tightness for a bounded set of Radon measures.

Definition 3 (i) A set $\Pi \subset \Gamma$ is *tight* if Π is bounded, and if for all $\varepsilon > 0$, there exists a finite number of compact projections $a_1, ..., a_n$ such that for all $\gamma \in \Pi$, $\gamma(1 - a_i) \leq \varepsilon$ for some i $(1 \leq i \leq n)$.

(ii) A net (γ_α) in Γ is *tight* if (γ_α) is eventually bounded, and if for all $\varepsilon > 0$, there exists a finite number of compact projections $a_1, ..., a_n$ such that eventually $\gamma_\alpha(1 - a_i) \leq \varepsilon$ for some i $(1 \leq i \leq n)$.

In the commutative case, if the bounded maxitive set-capacity γ is represented by the usc function f, then γ is tight if and only if $\{t; f(t) \geq s\}$ is compact for all $s > 0$. This remains true in the general case as shows the following.

Corollary 2 *Let γ be a bounded maxitive capacity and z the operator which represents γ. Then, γ is tight if and only if z is strongly q-usc. In particular, a closed projection (as a capacity) is tight if and only if it is compact.*

Proof. The proof of Theorem 2 shows that the two following properties hold:

(a) $p \leq E^z_{\gamma(p)}$ for all $p \in B \cup C$.

(b) $\gamma(E^z_{[0,\lambda[}) \leq \lambda$ for all $\lambda \in \mathbb{R}$.

We can suppose $1 \notin \mathcal{U}$. Let γ be tight and $s > 0$. By definition, there exists $a \in A$ such that $\gamma(1-a) = \sup\{\lambda \in \sigma(z); \forall\varepsilon > 0, (1-a)(E_{\lambda+\varepsilon} - E_{\lambda-\varepsilon}) \neq 0\} < s$. If $\gamma(1-a) = 0$, then $1 - a \leq E_0$ by (a), and thus $E_{[s',\infty[} \leq a$ for all $s' > 0$ and so $E_{[s',\infty[} \in A$. Assume $\gamma(1-a) \neq 0$. First suppose $s \in \sigma(z)$. Then, there exists $\varepsilon_a > 0$ such that $(1-a)(E_{s+\varepsilon_a} - E_{s-\varepsilon_a}) = 0$ with $E_{s+\varepsilon_a} - E_{s-\varepsilon_a} \neq 0$. By (a), $1 - a \leq E_{\gamma(1-a)} \leq E_{s+\varepsilon_a}$ and thus $1 - a \leq E_{s-\varepsilon_a}$, or equivalently $E_{]s-\varepsilon_a,\infty[} \leq a$, and so $E_{[s,\infty[} \leq a$. If $s \notin \sigma(z)$, since $\gamma(p) \in \sigma(z)$ for all $p \neq 0$ in $B \cup C$, $E_{[\gamma(1-a),\infty[} \in A$ by the preceding case, and $E_{[s,\infty[} \leq E_{[\gamma(1-a),\infty[}$ implies $E_{[s,\infty[} \in A$. Reciprocally, suppose $E_{[s,\infty[} \in A$ for all $s > 0$. By (b), $\gamma(E_{[0,s[}) \leq s$ with $E_{[0,s[} = 1 - E_{[s,\infty[}$. $\square$

The following example gives a Gleason-type theorem where additivity is replaced by maxitivity.

Example 1 If $\mathcal{U}$ is the C*-algebra of compact operators on a complex Hilbert space $\mathcal{H}$, then $\mathcal{U}^{**}$ is the algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on $\mathcal{H}$, compact projections are the finite dimensional ones and each projection in $\mathcal{B}(\mathcal{H})$ is open. A positive operator on $\mathcal{H}$ is q-usc and strongly q-lsc, and thus compact if and only if it is strongly q-usc (a general result ([2]) asserts that $x \in \mathcal{U}^{**}_h$ is strongly q-lsc and strongly q-usc if and only if $x \in \mathcal{U}_h$). Then, Theorem 2 gives a bijection between the set of completely maxitive maps on the set of all projections in $\mathcal{B}(\mathcal{H})$ to $[0,\infty[$, and the set of positive operators on $\mathcal{H}$ (notice that complete maxitivity for γ implies $\gamma(0) = 0$ with the general convention $\sup_\emptyset L = \inf L$ for any complete lattice L). A positive operator is compact if and only if it represents a tight maxitive capacity.

2 Vague compactness

We first recall some results in noncommutative topology. The Urysohn's lemma ([2], Lemma 3.1) states that if $a \in A$, $b \in B$ and $a \leq b$, then there exists $x \in \mathcal{U}_+$ such that $a \leq x \leq b$. It follows that $b' = E^x_{]2/3,1]} \in B$, $a' = E^x_{[1/3,1]} \in A$ with $a \leq b' \leq a' \leq b$, which gives the non commutative analogue of local compactness. If $a \in A$, $b \in B$ with $a \leq b$, then a is compact as an element of $b\mathcal{U}^{**}b$ ([4], Corollary 2.48), and if b' is open as an element of $b\mathcal{U}^{**}b$, then $b' \in B$ ([4], Prop. 2.14). By Urysohn's lemma, we obtain the non commutative Hausdorff property: for all $a_1 \in A$, $a_2 \in A$, $b \in B$ with $a_1a_2 = 0$ and $a_1 \vee a_2 \leq b$, there exists $b_1 \in B$, $b_2 \in B$ such that $a_1 \leq b_1 \leq b$, $a_2 \leq b_2 \leq b$ and $b_1b_2 = 0$.

By Theorem 1, the following theorem with $\Gamma_{a,[0,s]}$ $(s < \infty)$ is a noncommutative version of the vague compactness of the set of Radon measures on X with mass less than or equal to s ([3], Proposition 4.6).

Theorem 3 *If Δ is a compact subset of $[0,\infty]$ containing 0, then Γ_Δ, $\Gamma_{sa,\Delta}$ and $\Gamma_{a,\Delta}$ are vaguely compact Hausdorff spaces.*

Proof. We show first that Γ is a vaguely Hausdorff space. If $\gamma_1 \neq \gamma_2$ in Γ, and since any element in Γ is completely determined by its values on A, there exists $a \in A$ such that $\gamma_1(a) \neq \gamma_2(a)$, say $\gamma_1(a) < \gamma_2(a)$. There exists $b \in B$ and $\varepsilon > 0$ such that $b \geq a$ and $\gamma_1(b) < \gamma_1(a) + \varepsilon < \gamma_2(a)$. By Urysohn's lemma , there exists a' in A and b' in B

such that $a \le b' \le a' \le b$. Then, γ_1 is in the open set $V_1 = \{\delta \in \Gamma; \delta(a') < \gamma_1(a) + \varepsilon\}$ and γ_2 in the open set $V_2 = \{\delta \in \Gamma; \delta(b') > \gamma_1(a) + \varepsilon\}$ with $V_1 \cap V_2 = \emptyset$.

We show now that Γ_Δ is vaguely compact. Let

$$\mathcal{V} = \bigcup_{b_i \in B,\ x_i > 0,\ i \in I} \{\gamma \in \Gamma_\Delta; \gamma(b_i) > x_i\} \cup \bigcup_{a_j \in A,\ y_j > 0,\ j \in J} \{\gamma \in \Gamma_\Delta; \gamma(a_j) < y_j\}$$

be a (subbasic) open cover of Γ_Δ. By Alexander's subbase theorem it suffices to find a finite subcover of $\mathcal{V}$. Since $0 \in B$ we can suppose $x_i = 0$ for some $i \in I$, adding $\emptyset = \{\gamma \in \Gamma_\Delta; \gamma(0) > 0\}$ to $\mathcal{V}$. Replacing x_i by $\sup\{x \in \Delta; x \le x_i\}$, we can suppose $x_i \in \Delta$ for all $i \in I$. Define γ_0 on $A \cup B$ by:

(i) $\gamma_0(a) = \inf_{b_i \ge a, b_i \in B, i \in I}\ x_i$ for all $a \in A$,

(ii) $\gamma_0(b) = \sup_{a \le b, a \in A} \gamma_0(a)$ for all $b \in B$.

Since

$$\forall i \in I, \quad \gamma_0(b_i) = \sup_{a \le b_i, a \in A} \gamma_0(a) = \sup_{a \le b_i, a \in A} \{\inf_{b_k \ge a, b_k \in B, k \in I} x_k\} \le x_i$$

we obtain

$$\gamma_0(a) \le \inf_{b \ge a, b \in B} \gamma_0(b) \le \inf_{b_i \ge a, b_i \in B, i \in I} \gamma_0(b_i) \le \inf_{b_i \ge a, b_i \in B, i \in I} x_i = \gamma_0(a)$$

and so γ_0 is outer regular on A. Extend γ_0 to $B \cup C$ defining $\gamma_0(c) = \inf_{b \ge a, b \in B} \gamma_0(b)$ for all $c \in C$. Then, γ_0 is Δ-valued, $\gamma_0(0) = 0$, and so $\gamma_0 \in \Gamma_\Delta$.

One of the open sets of $\mathcal{V}$ contains γ_0 and since $\gamma_0(b_i) \le x_i$ for all $i \in I$, $\gamma_0(a_{j_0}) < y_{j_0}$ for some $j_0 \in J$. There exists $i_0 \in I$ such that $b_{i_0} \ge a_{j_0}$ and $x_{i_0} < y_{j_0}$. Now, let $\gamma \in \Gamma_\Delta$. If $\gamma \notin \{\delta \in \Gamma_\Delta; \delta(a_{j_0}) < y_{j_0}\}$, then $\gamma(b_{i_0}) \ge \gamma(a_{j_0}) \ge y_{j_0} > x_{i_0}$ and $\gamma \in \{\delta \in \Gamma_\Delta; \delta(b_{i_0}) > x_{i_0}\}$. So Γ_Δ is compact.

We show that Γ_{sa} is vaguely closed in Γ. Let $\gamma \in \Gamma \backslash \Gamma_{sa}$. There are $a_1 \in A$, $a_2 \in A$, $x_1 > 0$ and $x_2 > 0$ such that $\gamma(a_1) < x_1$, $\gamma(a_2) < x_2$ and $\sup_{a \le a_1 \vee a_2, a \in A} \gamma(a) > x_1 + x_2$. Let $\{a'_1, a'_2\} \subset A$, $\{b_1, b'_1, b_2, b'_2\} \subset B$ such that $a_1 \le b'_1 \le a'_1 \le b_1$, $a_2 \le b'_2 \le a'_2 \le b_2$, $\gamma(b_1) < x_1$ and $\gamma(b_2) < x_2$. Then, γ is in the open set $V = \{\delta \in \Gamma;\ \delta(a'_2) < x_2,\ \delta(a'_1) < x_1,\ \delta(b'_1 \vee b'_2) > x_1 + x_2\}$ which does not intersect Γ_{sa}.

We show that Γ_a is vaguely closed in Γ. Let $\gamma \in \Gamma_{sa} \backslash \Gamma_a$. There are $a_1 \in A$, $a_2 \in A$, $x_1 > 0$ and $x_2 > 0$ such that $a_1 a_2 = 0$, $\gamma(a_1 \vee a_2) < x_1 + x_2$, $\gamma(a_1) > x_1$, $\gamma(a_2) > x_2$. There exists $b \in B$ such that $b \ge a_1 \vee a_2$ and $\gamma(b) < x_1 + x_2$. By Hausdorff property, there are $b_1 \in B$, $b_2 \in B$ such that $a_1 \le b_1 \le b$, $a_2 \le b_2 \le b$ and $b_1 b_2 = 0$. Let $\{a'_1, a'_2\} \subset A$, $\{b'_1, b'_2\} \subset B$ such that $a_1 \le b'_1 \le a'_1 \le b_1$ and $a_2 \le b'_2 \le a'_2 \le b_2$. Therefore γ is in the open set $V = \{\delta \in \Gamma;\ \delta(b'_1) > x_1,\ \delta(b'_2) > x_2,\ \delta(a'_1 \vee a'_2) < x_1 + x_2\}$ which does not intersect Γ_a.

Since $\Gamma_{sa,\Delta} = \Gamma_{sa} \cap \Gamma_\Delta$ and $\Gamma_{a,\Delta} = \Gamma_a \cap \Gamma_\Delta$ the theorem is proved. □

The following corollary is a noncommutative version of the criterion of vague relative compactness in the set of Radon probability measures on X ([11], Proposition 2.5).

Corollary 3 *A set $\Pi \subset \Gamma_{a,1}$ is vaguely relatively compact in $\Gamma_{a,1}$ if and only if for all $\varepsilon > 0$, there exists a finite number of compact projections $a_1, ..., a_n$ such that: $\gamma \in \Pi$ implies $\gamma(a_k) > 1 - \varepsilon$ for some k $(1 \le k \le n)$.*

Proof. The condition is sufficient. By Theorem 3, it suffices to show that the vague closure of Π in Γ is in Γ_1. Let $\gamma \in \overline{\Pi}$ and (γ_α) be a net in Π converging to γ. If $\gamma(1) < 1$, then there exists $\varepsilon > 0$ such that $\gamma(a) < 1 - \varepsilon$ for all $a \in A$, and thus for all finite family of compact projections $\{a_1, ..., a_n\}$, we have eventually $\gamma_\alpha(a_k) < 1 - \varepsilon$ for all k $(1 \leq k \leq n)$, giving the contradiction. If $\gamma(1) > 1$, γ is in the open set $V = \{\delta \in \Gamma : \ \delta(1) > 1\}$ and eventually $\gamma_\alpha(1) > 1$ which is impossible. Thus, $\gamma(1) = 1$.

Conversely, let $\Pi \subset \Gamma_{a,1}$ be vaguely relatively compact in $\Gamma_{a,1}$, and $\varepsilon > 0$. For each $\gamma \in \overline{\Pi} \cap \Gamma_{a,1}$, there exists $a'_\gamma \in A$, $a_\gamma \in A$, $b_\gamma \in B$, such that $a'_\gamma \leq b_\gamma \leq a_\gamma \leq 1$ and $\gamma(a'_\gamma) > 1 - \varepsilon$. When γ ranges over $\overline{\Pi} \cap \Gamma_{a,1}$, the open sets $V_\gamma = \{\delta \in \overline{\Pi} \cap \Gamma_{a,1}; \delta(b_\gamma) > 1 - \varepsilon\}$ cover the compact set $\overline{\Pi} \cap \Gamma_{a,1}$. Thus, there exists a finite subcover $V_{\gamma_1} \cup ... \cup V_{\gamma_n}$ of $\overline{\Pi} \cap \Gamma_{a,1}$, and the condition follows with $a_{\gamma_1}, ..., a_{\gamma_n}$. □

The following corollary is a noncommutative version of the criterion of vague relative compactness in the set of Radon measures on X ([11], Proposition 2.4).

Corollary 4 *A set* $\Pi \subset \Gamma_{a,<}$ *is vaguely relatively compact in* $\Gamma_{a,<}$ *if and only if*

$$\sup_{\gamma \in \Pi} \ \gamma(a) < \infty \qquad \text{for all } a \in A.$$

Proof. The condition is sufficient. By Theorem 3, it suffices to show that $\overline{\Pi} \subset \Gamma_<$. Let $\gamma \in \overline{\Pi}$ and (γ_α) be a net in Π converging to γ. If $\gamma \notin \Gamma_<$, then there are $a \in A$, $a' \in A$, $b' \in B$ such that $a \leq b' \leq a' \leq 1$ and $\gamma(a) = +\infty$. By hypothesis there exists $M > 0$ such that $\sup_{\gamma \in \Pi} \ \gamma(a') < M$ and so, $\sup_{\gamma \in \Pi} \ \gamma(b') < M$. But the open set $V = \{\delta \in \Gamma : \ \delta(b') > M\}$ contains γ and eventually γ_α, which gives the contradiction.

Conversely, let Π be relatively compact in $\Gamma_{a,<}$, and $a \in A$. Then, $\bigcup_{M>0} V_M$ where $V_M = \{\delta \in \overline{\Pi} \cap \Gamma_{a,<}; \delta(a) < M\}$ is an open cover of the compact set $\overline{\Pi} \cap \Gamma_{a,<}$ and so there is a finite subfamily covering $\overline{\Pi} \cap \Gamma_{a,<}$, namely $\bigcup_{1 \leq i \leq n} \ V_{M_i} = V_{M_{max}}$ where $M_{max} = \max_{1 \leq i \leq n} \ M_i$. □

3 Narrow compactness

Let T be a topological space and $S \subset T$. We say that S is *net-compact* in T if S satisfies one of the following equivalent properties ([14]):

(i) Each net on S has a subnet converging to some point in T.

(ii) Each open cover of T has a finite subfamily covering S.

Clearly relative compactness is stronger than net-compactness, and it is well known that both notions coincide when T is regular.

The following proposition shows that in Γ, narrow net-compactness and narrow relative compactness are equivalent for σ-unital C*-algebras.

Proposition 2 *If* $\mathcal{U}$ *is* σ*-unital, then* Γ *is a narrowly regular space.*

Proof. Let F be a closed subbasic set of the form $F = \{\delta \in \Gamma; \delta(c) \geq y\}$ $(c \in C, y > 0)$ not containing γ. Then $\gamma(c) < y$, and by outer regularity, there exists $b \in B$, $c \leq b$, $\varepsilon > 0$ such that $\gamma(b) < y - \varepsilon$. Since $\mathcal{U}$ is σ-unital, there exists $c' \in C$, $b' \in B$ such that $c \leq b' \leq c' \leq b$ ([4], Corollary 3.32). Then $\gamma \in V_\gamma = \{\delta \in \Gamma; \delta(c') < y - \varepsilon\}$, $F \subset V_F = \{\delta \in \Gamma; \delta(b') > y - \varepsilon\}$ and V_γ and V_F are two open sets with $V_\gamma \cap V_F = \emptyset$.

Let F be a closed subbasic set of the form $F = \{\delta \in \Gamma; \delta(b) \leq x\}(b \in B, x > 0)$ not containing γ. Then $\gamma(b) > x$, and by inner regularity, there exists $a \in A$, $a' \in A$, $b' \in B$, $\varepsilon > 0$ such that $a \leq b' \leq a' \leq b$ and $\gamma(a) > x+\varepsilon$. Then $\gamma \in V_\gamma = \{\delta \in \Gamma; \delta(b') > x+\varepsilon\}$, $F \subset V_F = \{\delta \in \Gamma; \delta(a') < x+\varepsilon\}$ and V_γ and V_F are two open sets with $V_\gamma \cap V_F = \emptyset$. □

Definition 4 (i) A set $\Pi \subset \Gamma$ is *uniformly inner regular with respect to* C, if for all $c \in C$, for all $\varepsilon > 0$, for all $b \in B$ with $b \geq c$, there exists a finite number of compact projections $a_1, ..., a_n$ with $a_i \leq b$ $(1 \leq i \leq n)$ such that if $\gamma \in \Pi$, then $\gamma(c) \leq \gamma(a_i) + \varepsilon$ for some i $(1 \leq i \leq n)$.

(ii) A net (γ_α) in Γ is *uniformly inner regular with respect to* C if for all $c \in C$, for all $\varepsilon > 0$, for all $b \in B$ with $b \geq c$, there exists a finite number of compact projections $a_1, ..., a_n$ with $a_i \leq b$ $(1 \leq i \leq n)$ such that eventually $\gamma_\alpha(c) \leq \gamma_\alpha(a_i) + \varepsilon$ for some i $(1 \leq i \leq n)$.

It is easy to see that a bounded set Π of Radon measures on X is uniformly inner regular with respect to $\mathcal{F}$ if and only if Π is tight. Moreover, Π is tight if and only if the narrow closure $\overline{\Pi}$ is tight. By Theorem 1, the following theorem (with $\Gamma_{a,[0,\infty[}$ and $\Gamma_{a,1}$) is a noncommutative version of Prohorov's theorems: a set Π of bounded Radon measures (resp. probability measures) is narrowly relatively compact if and only if Π is tight.

Theorem 4 *A set* $\Pi \subset \Gamma_{[0,\infty[}$ *is narrowly net compact in* $\Gamma_{[0,\infty[}$ *if and only if* Π *is bounded and uniformly inner regular with respect to* C*. The same is true with* $\Gamma_{sa,[0,\infty[}$*,* $\Gamma_{a,[0,\infty[}$ *and* $\Gamma_{a,1}$ *in place of* $\Gamma_{[0,\infty[}$*.*

Proof. Suppose that $\Pi \subset \Gamma_{[0,\infty[}$ is narrowly net compact in $\Gamma_{[0,\infty[}$, and let $c \in C$, $\varepsilon > 0$, $b \in B$ with $b \geq c$. For each $\gamma \in \Gamma_{[0,\infty[}$, there exists $a_\gamma \in A$, $a'_\gamma \in A$, $b'_\gamma \in B$ such that $a_\gamma \leq b'_\gamma \leq a'_\gamma \leq b$ and $\gamma(c) < \gamma(a_\gamma) + \varepsilon \leq \gamma(b'_\gamma) + \varepsilon \leq \gamma(a'_\gamma) + \varepsilon \leq \gamma(b) + \varepsilon$. Then, each $\gamma \in \Gamma_{[0,\infty[}$ is in the open set $W_\gamma = V_{1,\gamma} \cap V_{2,\gamma}$, where $V_{1,\gamma} = \{\delta \in \Gamma_{[0,\infty[}; \delta(c) < \gamma(a_\gamma) + \varepsilon\}$ and $V_{2,\gamma} = \{\delta \in \Gamma_{[0,\infty[}; \delta(b'_\gamma) > \gamma(a_\gamma) - \varepsilon\}$. When γ ranges over $\Gamma_{[0,\infty[}$, the sets W_γ form an open covering of $\Gamma_{[0,\infty[}$, and since Π is net compact, there exists a finite number $W_{\gamma_1}, ..., W_{\gamma_n}$ which covers Π. That is there exists a finite number of $a_{\gamma_i} \in A$, $a'_{\gamma_i} \in A$, $b'_{\gamma_i} \in B$ with $a_{\gamma_i} \leq b'_{\gamma_i} \leq a'_{\gamma_i} \leq b$ $(1 \leq i \leq n)$ such that if $\gamma \in \Pi$, then $\gamma(c) < \gamma_k(a_{\gamma_k}) + \varepsilon < \gamma(b'_{\gamma_k}) + 2\varepsilon \leq \gamma(a'_{\gamma_k}) + 2\varepsilon$ for some k $(1 \leq k \leq n)$. Taking $c = 1$, this shows that Π is bounded by $\sup_{1 \leq i \leq n} \gamma_i(a_{\gamma_i})$.

Conversely, suppose that $\Pi \subset \Gamma_{[0,\infty[}$ is bounded and uniformly inner regular with respect to C. Let (γ_α) be a universal net in Π, $c \in C$, and $\varepsilon > 0$. By Theorems 3, (γ_α) converges vaguely to some $\gamma \in \Gamma_{[0,\infty[}$ (since $\gamma(1) \leq \sup_{\delta \in \Pi} \delta(1)$). There exists $b \geq c$ such that $\gamma(b) < \gamma(c) + \varepsilon$, and since the net is universal, we have eventually $\gamma_\alpha(c) \leq \gamma_\alpha(a) + \varepsilon$ for some $a \in A$ with $a \leq b$. By vague convergence, we have eventually $\gamma_\alpha(c) \leq \gamma_\alpha(a) + \varepsilon < \gamma(a) + 2\varepsilon \leq \gamma(b) + 2\varepsilon < \gamma(c) + 3\varepsilon$, and (γ_α) converges narrowly to γ.

By Theorem 3, $\Gamma_{sa,[0,\infty[}$ and $\Gamma_{a,[0,\infty[}$ are vaguely closed subsets of $\Gamma_{[0,\infty[}$, and $\Gamma_{a,1}$ is narrowly closed in $\Gamma_{a,[0,\infty[}$ (if $\gamma \in \Gamma_{a,[0,\infty[} \backslash \Gamma_{a,1}$, then $\gamma(1) < 1$ or $\gamma(1) > 1$, and in either case we have an open neighborhood of γ which does not intersect $\Gamma_{a,1}$). This proves the second assertion. □

The proof of Theorem 4 gives the following version for nets:

Theorem 5 *If (γ_α) is a net in $\Gamma_{[0,\infty[}$ converging vaguely to $\gamma \in \Gamma_{[0,\infty[}$, then (γ_α) converges narrowly to γ if and only if (γ_α) is uniformly inner regular with respect to C.*

4 Large deviations

Let (t_α) be a net in $]0,\infty[$ converging to 0. Recall that a net of Radon probability measures (μ_α) on X satisfies a vague (resp. narrow) large deviation principle with powers (t_α) if there exists a positive usc function f on X such that:

(i) $\limsup \mu_\alpha(Y)^{t_\alpha} \leq \sup_{t\in Y} f(t)$ for all $Y \in \mathcal{K}$ (resp. $\mathcal{F}$),

(ii) $\liminf \mu_\alpha(G)^{t_\alpha} \geq \sup_{t\in G} f(t)$ for all $G \in \mathcal{G}$.

We say that f is the governing function (a large deviation principle is often given in a logarithmic form, with the so-called rate function $-\log f$). The observation preceding Theorem 2 allows us to extend this definition to nets of capacities on $\mathcal{U}$.

Definition 5 Let (t_α) be a net in $]0,\infty[$ converging to 0. A net (γ_α) in Γ satisfies a vague (resp. narrow) *large deviation principle* with powers (t_α) if there exists a positive q-usc operator z such that:

(i)

$$\limsup \gamma_\alpha(p)^{t_\alpha} \leq \sup\{\lambda \in \sigma(z); \forall \varepsilon > 0, pE^z_{]\lambda-\varepsilon,\lambda+\varepsilon]} \neq 0\}$$

for all $p \in A$ (resp. C),

(ii)

$$\liminf \gamma_\alpha(b)^{t_\alpha} \geq \sup\{\lambda \in \sigma(z); \forall \varepsilon > 0, bE^z_{]\lambda-\varepsilon,\lambda+\varepsilon]} \neq 0\}$$

for all $b \in B$.

We say that the operator z governs the large deviation principle.

Since for each $\gamma \in \Gamma$ and $t > 0$, γ^t defined by $\gamma^t(p) = \gamma(p)^t$ for all $p \in B \cup C$ is a capacity, we obtain by Theorem 2 the following equivalent definition.

Theorem 6 *A net (γ_α) in Γ satisfies a vague (resp. narrow) large deviation principle with powers (t_α) if and only if there exists a bounded maxitive capacity γ such that $(\gamma_\alpha^{t_\alpha})$ converges vaguely (resp. narrowly) to γ. The positive q-usc operator representing γ governs the large deviation principle.*

Now, we can use the topological properties of Γ to extend some results in large deviations theory. Since Γ is vaguely Hausdorff, by Theorem 6 we obtain the following.

Proposition 3 *A net of capacities satisfying a vague large deviation principle is governed by a unique positive q-usc operator.*

Suppose that $\mathcal{U} = \mathcal{C}_0(X)$, and let (μ_α) be a net of Radon probability measures on X, and (t_α) a net in $]0,\infty[$ converging to 0. The tightness of $(\mu_\alpha^{t_\alpha})$ implies the uniform inner regularity on $\mathcal{F}$ (and a fortiori with respect to $\mathcal{F}$): indeed, since $\mathcal{U}$ is commutative we can suppose $n = 1$ in Definition 3 of tightness; let $F \in \mathcal{F}$, $\varepsilon > 0$ and $K \in \mathcal{K}$ such that eventually $\mu_\alpha(X \backslash K)^{t_\alpha} \leq \varepsilon$. Since eventually $t_\alpha \leq 1$, we obtain eventually

$$\mu_\alpha(F)^{t_\alpha} \leq \mu_\alpha(F \cap K)^{t_\alpha} + \mu_\alpha(F \cap (X \backslash K))^{t_\alpha} \leq \mu_\alpha^{t_\alpha}(F \cap K) + \varepsilon.$$

When $t_\alpha = \alpha$ ($\alpha > 0$), the tightness of $(\mu_\alpha^{t_\alpha})$ is known as the exponential tightness of the family $\{\mu_\alpha; \alpha > 0\}$. Thus, the following Theorem 7 is a noncommutative version of the well known result ([9], Lemma 2.1.5): if an exponentially tight family $\{\mu_\alpha; \alpha > 0\}$ of Radon probability measures on X satisfies a vague large deviation principle with powers $\{\alpha; \alpha > 0\}$ and governing function f, then it satisfies a narrow large deviation principle with same powers and same governing function with $\{t; f(t) \geq s\}$ compact for all $s > 0$.

Theorem 7 *Let (γ_α) be a net in Γ satisfying a vague large deviation principle with powers (t_α). If $(\gamma_\alpha^{t_\alpha})$ is tight, then the large deviation principle is governed by a positive strongly q-usc operator. The net (γ_α) satisfies a narrow large deviation principle with same powers and same governing operator if and only if $(\gamma_\alpha^{t_\alpha})$ is uniformly inner regular with respect to C.*

Proof. Let $\gamma \in \Gamma_{m,[0,\infty[}$ such that $(\gamma_\alpha^{t_\alpha})$ converges vaguely to γ, and $(\gamma_\beta^{t_\beta})$ be a universal subnet of $(\gamma_\alpha^{t_\alpha})$. If $(\gamma_\alpha^{t_\alpha})$ is tight, then for all $\varepsilon > 0$, we have eventually $\gamma_\beta^{t_\beta}(1 - a) \leq \varepsilon$ for some $a \in A$, and so γ is tight. By Corollary 2, the positive q-usc operator representing γ is strongly q-usc. By Theorem 5, in order to prove the last assertion it suffices to show the following: if $(\gamma_\alpha^{t_\alpha})$ is inner regular with respect to C, then $(\gamma_\beta^{t_\beta})$ is eventually in $\Gamma_{[0,\infty[}$. Otherwise, we have eventually $\gamma_\beta^{t_\beta}(1) = \infty$, and by uniform inner regularity with respect to 1, for all $\varepsilon > 0$, there exists $a \in A$ such that eventually $\gamma_\beta^{t_\beta}(1) \leq \gamma_\beta^{t_\beta}(a) + \varepsilon$. We obtain eventually, $\gamma_\beta^{t_\beta}(a) = \infty$ for some $a \in A$, and by vague convergence $\gamma(a) = \infty$ which is impossible. □

4.1 The case where $\mathcal{U}$ is a full matrix algebra

Let $\mathcal{U} = M_n(\mathbf{C})$ be the full $n \times n$-matrix algebra. In this case $\mathcal{U} = \mathcal{U}^{**}$, each projection in $M_n(\mathbf{C})$ is open and compact, a capacity on $M_n(\mathbf{C})$ is just a $[0, \infty]$-valued map on the set B of all projections of $M_n(\mathbf{C})$ vanishing on 0, and the vague and narrow topologies coincide. Therefore, a net (ω_α) of states on $M_n(\mathbf{C})$ satisfies a large deviation principle with powers (t_α) if and only if there exists a completely maxitive map γ on B such that

$$\forall p \in B, \qquad \lim \omega_\alpha^{t_\alpha}(p) = \gamma(p).$$

In [6] we proved that when the limit $\gamma(p) = \lim \omega_\alpha^{t_\alpha}(p)$ exists for all $p \in B$, then γ is necessarily completely maxitive, and so the existence of large deviations is equivalent to the existence of this limit. By applying Theorem 3 and Theorem 6 we obtain the following noncommutative version of a well known fact in classical large deviations theory: any net (μ_α) of Radon probability measures on X has a subnet (μ_β) satisfying a vague large deviation principle with powers (t_β).

Proposition 4 *Any net (ω_α) of states on $M_n(\boldsymbol{C})$ has a subnet (ω_β) satisfying a large deviation principle with powers (t_β). More precisely, there exists a positive operator $z \in M_n(\boldsymbol{C})$ such that*

$$\lim \omega_\beta^{t_\beta}(p) = \max\{\lambda \in \sigma(z); pE_{\{\lambda\}}^z \neq 0\}$$

for all projections $p \in M_n(\boldsymbol{C})$.

Example 2 Let $(\mathcal{T}_t)_{t\geq 0}$ be the Markov quantum semigroup on $M_2(\mathbf{C})$ given by the values of its generator $\mathcal{L}$ on the Pauli matrices $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, and $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$:

(i) $\mathcal{L}(\sigma_x) = -\frac{\eta}{2}\sigma_x - 2\xi\sigma_y$,

(ii) $\mathcal{L}(\sigma_y) = 2\xi\sigma_x - \frac{\eta}{2}\sigma_y$,

(iii) $\mathcal{L}(\sigma_z) = -\eta(1+\sigma_z)$,

with $\eta > 0$ and $\xi \geq 0$ (this semigroup has been considered in [8], and we refer to that paper for the physical interpretation). Direct calculations show that $\rho_\infty = (\frac{1-\sigma_z}{2})$ is the unique pure invariant state.

Proposition 5 *For each initial state* $\rho \neq \rho_\infty$*, the net of states given by the predual semigroup* $(\mathcal{T}_{*t}(\rho))_{t\geq 0}$ *satisfies a large deviation principle with powers* $(1/t)$ *and governing operator*

$$e^{-\eta}(1-\rho_\infty) + \rho_\infty.$$

Proof. Let ρ be a state on $M_2(\mathbf{C})$ with $\rho \neq \rho_\infty$, and let ω_t denote the state $\mathcal{T}_{*t}(\rho)$ for all $t \geq 0$. By Theorem 6 and Proposition 4, it suffices to prove that if a subnet $(\omega_{t_\alpha}^{1/t_\alpha})$ satisfies a large deviation principle with governing operator z, then $z = e^{-\eta}(1-\rho_\infty) + \rho_\infty$. Let γ_z denote the maxitive capacity represented by z. Notice that if there exists some projection p such that $0 < \gamma_z(p) < 1$, then $z = \gamma_z(p)p + (1-p)$ (indeed, $p \neq 0$ and $p \neq 1$ by hypothesis, and since $\gamma_z(E^z_{\{\gamma_z(p)\}}) = \gamma_z(p)$ we obtain $E^z_{\{\gamma_z(p)\}} \neq 0$ and $E^z_{\{\gamma_z(p)\}} \neq 1$; if $E^z_{\{\gamma_z(p)\}} \neq p$, then $\gamma_z(E^z_{\{\gamma_z(p)\}} \vee p) = \gamma_z(1) = 1$ which is impossible by maxitivity of γ_z; it follows that $E^z_{\{\gamma_z(p)\}} = p$ and $z = \gamma_z(p)p + (1-p)$). Therefore, it suffices to show that if a subnet $\omega_{t_\alpha}^{1/t_\alpha}(1-\rho_\infty)$ has a limit, then this limit is $e^{-\eta}$. By (iii) we obtain easily

$$\omega_t(\sigma_z) = e^{-\eta t}(1 + \mathrm{tr}\ \rho\sigma_z) - 1$$

and so

$$\omega_t(1-\rho_\infty) = \omega_t(\frac{1+\sigma_z}{2}) = e^{-\eta t}\frac{(1+\mathrm{tr}\ \rho\sigma_z)}{2}.$$

Since $1 + \mathrm{tr}\ \rho\sigma_z = 0$ if and only if $\rho = \rho_\infty$, we obtain

$$\lim_{t\to\infty} \omega_t^{1/t}(1-\rho_\infty) = e^{-\eta}.$$

□

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