

# Around Hilbert 16-th Problem

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# Introduction

The present text is the lecture notes of a mini-course given at Santiago-Chile. Its objective is to introduce the reader with some problems arising from the centennial Hilbert 16-th problem, H16 for short. Our aim is not to collect all the developments and theorems in direction of H16 (for this see for instance [9]), but to present a way of breaking the problem in many pieces and observing the fact that even such partial problems are extremely difficult to treat. Our point of view is algebraic and we want to point out that the both real and complex algebraic geometry would be indispensable for a systematic approach to the H16. Here some notations that we use:

- For a topological space  $X$  and  $x \in X$  we denote by  $(X, x)$  an small neighborhood of  $x$  in  $X$ .
- We will use the fields  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{C}$ . We denote by  $\mathbb{K}[x, y]$  the ring of polynomials in  $x, y$  with coefficients in  $\mathbb{K}$ .
- The set of polynomial differential 1-forms

$$\Omega_{\mathbb{K}^2}^1 := \{Pdy - Qdx \mid P, Q \in \mathbb{K}[x, y]\},$$

and differential two forms

$$\Omega_{\mathbb{K}^2}^2 = \{Pdx \wedge dy \mid P \in \mathbb{K}[x, y]\}.$$

- We will use  $X_{\mathbb{R}}$  and  $X_{\mathbb{C}}$  to distinguish between real and complex objects.

# Chapter 1

## Polynomial differential equation

What we want to study is the following ordinary differential equation:

$$(1.1) \quad \begin{cases} \dot{x} = P(x, y) \\ \dot{y} = Q(x, y) \end{cases},$$

where  $P, Q$  are two polynomials in  $x$  and  $y$  with coefficients in  $\mathbb{R}$  and  $\dot{x} = \frac{dx}{dt}$ . We will assume that  $P$  and  $Q$  do not have common factors. Its solutions are the trajectories of the vector field:

$$X := P(x, y) \frac{\partial}{\partial x} + Q(x, y) \frac{\partial}{\partial y}$$

(we will also write  $X = (P, Q)$ ). Let us first recall the first basic theorem of ordinary differential equations.

**Theorem 1.1.** *For  $A \in \mathbb{R}^2$  if  $X(A) \neq 0$  then there is a unique analytic function*

$$\lambda : (\mathbb{R}, 0) \rightarrow \mathbb{R}^2$$

such that

$$\gamma(0) = A, \quad \dot{\gamma} = X(\gamma(t))$$

*Proof.* Let us write formally

$$\gamma = \sum_{i=0}^{\infty} \gamma_i t^i, \quad \gamma_i \in \mathbb{R}^2, \quad \gamma_0 := A$$

and substitute it in  $\dot{\gamma} = X(\gamma)$ . It turns out that if  $X(A) \neq 0$  then  $\gamma_i$  can be written in a unique way in terms of  $\gamma_j$ ,  $j < i$ . This guaranties the existence of a unique formal  $\gamma$ .  $\square$

**Exercise 1.1.** Recover the proof of convergence of  $\gamma$  from classical books on ordinary differential equations.

**Example 1.1.** The trajectories of the differential equation

$$(1.2) \quad \begin{cases} \dot{x} = 2y + \frac{x^2}{2} \\ \dot{y} = 3x^2 - 3 + 0.9y \end{cases}$$

are depicted in Figure (1.1).

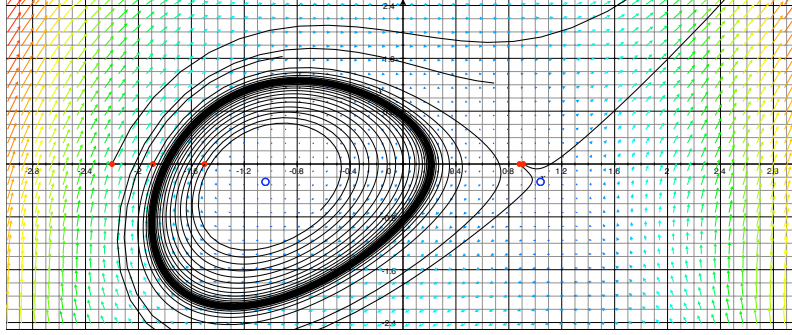


Figure 1.1: A limit cycle crossing  $(x, y) \sim (-1.79, 0)$

The collection of the images of the solutions of (1.1) gives us an analytic singular foliation  $\mathcal{F} = \mathcal{F}(X)_{\mathbb{R}} = \mathcal{F}(X) = \mathcal{F}_{\mathbb{R}}$  in  $\mathbb{R}^2$ . Therefore, when we are talking about a foliation we are not interested in the parametrization of its leaves (trajectories). It is left to the reader to verify that:

**Exercise 1.2.** For a polynomial  $R \in \mathbb{R}[x, y]$  the foliation associated to  $X$  and  $RX$  are the same.

For this reason from the beginning we have assumed that  $P$  and  $Q$  have no common factors. Being interested only on the foliation  $\mathcal{F}(X)$ , we may write (1.1) in the form

$$\frac{dy}{dx} = \frac{P(x, y)}{Q(x, y)},$$

$$\omega = 0, \quad \text{where } \omega = Pdy - Qdx \in \Omega_{\mathbb{R}^2}^1.$$

In the second case we also use the notation  $\mathcal{F} = \mathcal{F}(\omega)_{\mathbb{R}} = \mathcal{F}(\omega)$ .

**Definition 1.1.** The singular set of the foliation  $\mathcal{F}(Pdy - Qdx)$  is defined in the following way:

$$\text{Sing}(\mathcal{F}) = \text{Sing}(\mathcal{F})_{\mathbb{R}} := \{(x, y) \in \mathbb{R}^2 \mid P(x, y) = Q(x, y) = 0\}.$$

By our assumption  $\text{Sing}(\mathcal{F})$  is a finite set of points. The leaves of  $\mathcal{F}$  near a point  $A \in \text{Sing}(\mathcal{F})$  may be complicated.

**Exercise 1.3.** Using a software which draws the trajectories of vector fields, describe the solutions of (1.2) near its singularities.

## 1.1 Poincaré first return map

From topological point of view a leaf  $L$  of  $\mathcal{F} = \mathcal{F}(\omega)$  is either homeomorphic to  $\mathbb{R}$  or to the circle  $S^1 := \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ . In the second case  $L$  is called a closed solution of  $\mathcal{F}$  (but not yet a limit cycle). We consider a point  $p \in L$  and a transversal section  $\Sigma$  to  $\mathcal{F}$  at  $p$ . For any point  $q$  in  $\Sigma$  near enough to  $p$ , we can follow the leaf of  $\mathcal{F}$  in the anti-clockwise direction and since  $L$  is closed we will encounter a new point  $h(q) \in \Sigma$ . We have obtained an analytic function

$$h : \Sigma \rightarrow \Sigma,$$

which is called the Poincaré first return map. Later in the context of holomorphic foliations we will call it the holonomy map. Usually we take a coordinate system  $z$  in  $\Sigma$  with  $z(p) = 0$  and write the power series of  $h$  at 0:

$$h(z) = \sum_{i=0}^{\infty} \frac{h^{(i)}(0)}{i!} z^i$$

**Definition 1.2.**  $h'(0)$  is called the multiplier of the closed solution  $L$ . If the multiplier is 1 then we say that  $h$  is tangent to the identity. In this case the tangency order is  $n$  if

$$h^{(i)}(0) = 0, \quad h^{(n)}(0) \neq 0.$$

A closed solution  $L$  of  $\mathcal{F}$  is called a limit cycle if its Poincaré first return map is not identity. In case the Poincaré first return map is identity then the leaves of  $\mathcal{F}$  near  $L$  are also closed. In this case we can talk about the continuous family of cycles  $\delta_z$ ,  $z \in \Sigma$ , where  $\delta_z$  is the leaf of  $\mathcal{F}$  through  $z$ .

**Exercise 1.4.** Prove that the multiplier and order of tangency do not depend on the coordinate system  $z$  in  $\Sigma$ .

## 1.2 Hilbert 16-th problem

It is natural to ask whether a foliation  $\mathcal{F}(Pdy - Qdx)$  has a finite number of limit cycles. This is in fact the first part of Hilbert 16-th problem:

**Theorem 1.2.** *Each polynomial foliation  $\mathcal{F}(Pdy - Qdx)$  has a finite number of limit cycles.*

The above theorem was proved by Yu. Ilyashenko and J. Ecalle independently around 80's. We have associated to each foliations  $\mathcal{F}$  the number  $N(\mathcal{F})$  of its limit cycles of  $\mathcal{F}$ . It is natural to ask how  $N(\mathcal{F})$  depends on the ingredient polynomial  $P$  and  $Q$  of  $\mathcal{F}$ .

**Problem 1.1.** (Hilbert 16'th problem) Fix a natural number  $n \in \mathbb{N}$ . Is there some natural number  $N(n) \in \mathbb{N}$  such that each foliation  $\mathcal{F}(Pdx - Qdy)$  with  $\deg(P), \deg(Q) \leq n$  has at most  $N(n)$  limit cycles.

Of course, it would be of interest to give an explicit description of  $N(n)$  and more strongly determine the nature of

$$N(n) := \max\{N(\mathcal{F}(\omega)) \mid \omega = Pdy - Qdx, \deg(P), \deg(Q) \leq n\}.$$

One of the objective of the present text is to explain the fact that Hilbert 16'th problem is a combination of many unsolved difficult problems. We note that even the case  $n = 2$  is open.

## 1.3 Stable and unstable limit cycle

For a foliation  $\mathcal{F} = \mathcal{F}(Pdy - Qdx)$  let us define the affine degree

$$\deg(\mathcal{F}) = \max\{\deg(P), \deg(Q)\}.$$

We denote by  $\mathcal{F}(d)$  the space of degree  $d$  foliations. As we remarked before, for a constant number  $c \in \mathbb{R}$  the foliations  $\mathcal{F}(c\omega)$  and  $\mathcal{F}(\omega)$  are the same. Therefore,  $\mathcal{F}(d)$  is some open set in the projectivization of the coefficients space of  $P$  and  $Q$ .

We take a point  $\mathcal{F} \in \mathcal{F}(d)$  and ask what happens to the limit cycles of  $\mathcal{F}$  if perturb the coefficients of  $\mathcal{F}$ .

**Definition 1.3.** A limit cycle  $\delta$  of  $\mathcal{F} \in \mathcal{F}(d)$  is called to be stable if any perturbation of  $\mathcal{F}' \in \mathcal{F}(d)$  of  $\mathcal{F}$  has a limit cycle near to  $\delta$ . Otherwise, it is called unstable.

Let us take a perturbation  $\mathcal{F}_t \in \mathcal{F}(d)$ ,  $t \in (\mathbb{R}^n, 0)$ ,  $\mathcal{F}_0 = \mathcal{F}$ . Here,  $t$  can be chosen from the coefficient space of  $P$  and  $Q$ .

**Example 1.2.** For our example (1.2) we will use the following perturbation:

$$(1.3) \quad \mathcal{F}_t : \begin{cases} \dot{x} = 2y + \epsilon \frac{x^2}{2} \\ \dot{y} = 3x^2 - 3 + \epsilon sy \end{cases}, t := (\epsilon, s - 1) \in (\mathbb{R}^2, 0).$$

We take again a transversal section  $\Sigma$  to  $\mathcal{F}$  in  $p \in \delta$ . It turns out that  $\Sigma$  is also transversal to the perturbed foliations  $\mathcal{F}_t$ ,  $t \in (\mathbb{R}^n, 0)$ . Therefore, we can follow a leaf of  $\mathcal{F}_t$  through  $z \in \Sigma$  in the anti-clockwise direction and then return to  $\Sigma$  in a new point. In this way, we obtain the perturbed Poincaré return map

$$h : \Sigma \times (\mathbb{R}^n, 0) \rightarrow \Sigma \times (\mathbb{R}^n, 0), h(z, t) = (h_t(z), t).$$

**Proposition 1.1.** For a limit cycle  $\delta$  if

1. the multiplier  $\delta$  of  $\mathcal{F}$  is not equal to one or;
2. the multiplier  $\delta$  of  $\mathcal{F}$  is equal to one and its tangency order is odd,

then  $\delta$  is stable.

**Exercise 1.5.** It seems to be classical that the inverse of the above proposition is also true. This is equivalent to say that if the multiplier of a limit cycle  $\delta$  of  $\mathcal{F}$  is equal to one and its tangency order is even then there is a perturbation  $\mathcal{F}_t \in \mathcal{F}(d)$  of  $\mathcal{F}$  in which  $\delta$  disappears. Prove or disprove this.

# Chapter 2

## Holomorphic foliations

In this chapter we will do two main things. First, we will consider the foliation  $\mathcal{F}(\omega)$  in  $\mathbb{C}^2$  instead of  $\mathbb{R}^2$ . Second, we will compactify the ambient space  $\mathbb{C}^2$  into  $\mathbb{P}^2$ . This will be the beginning of the theory of holomorphic foliations on complex manifolds.

### 2.1 Complexification

The basic theorem of ordinary differential equations in the complex context can be written in the following way:

**Theorem 2.1.** *Let  $P, Q \in \mathbb{C}[x, y]$  and  $X = (P, Q)$ . For  $A \in \mathbb{C}^2$  if  $X(A) \neq 0$  then there is a unique holomorphic function*

$$\lambda : (\mathbb{C}, 0) \rightarrow \mathbb{C}^2$$

such that

$$\gamma(0) = A, \quad \dot{\gamma} = X(\gamma(t))$$

The proof is similar to the real case. The images of the complex solutions of the vector field  $X$  give us a holomorphic foliation  $\mathcal{F} = \mathcal{F}(\omega)_{\mathbb{C}} = \mathcal{F}_{\mathbb{C}}$  in  $\mathbb{C}^2$ . The leaves of  $\mathcal{F}_{\mathbb{C}}$  are two dimensional real manifolds embedded in a real four dimensional space. If  $P, Q \in \mathbb{R}[x, y]$  we will denote by  $\mathcal{F}_{\mathbb{R}} = \mathcal{F}(\omega)_{\mathbb{R}}$  the corresponding real foliation in  $\mathbb{R}^2$ . Note that  $\mathbb{R}^2 \subset \mathbb{C}^2$  and

$$\mathcal{F}_{\mathbb{R}} = \mathbb{R}^2 \cap \mathcal{F}_{\mathbb{C}}$$

i.e. the intersection of a leaf of  $\mathcal{F}_{\mathbb{C}}$  with  $\mathbb{R}^2$  is a union of leaves of  $\mathcal{F}_{\mathbb{R}}$ . Note that  $\mathcal{F}_{\mathbb{C}}$  may have more singularities.

**Example 2.1.** The holomorphic foliation  $\mathcal{F}_d$  defined in  $\mathbb{C}^2$  by the 1-form

$$\omega := (y^d - x^{d+1})dy - (1 - x^d y)dx$$

is called the Jouanolou foliation of degree  $d$ . Consider the group

$$G := \{\epsilon \in \mathbb{C} \mid \epsilon^{d^2+d+1} = 1\}.$$

It acts on  $\mathbb{C}^2$  discontinuously in the following way:

$$(\epsilon, (x, y)) \rightarrow (\epsilon^{d+1}x, \epsilon y) \quad \epsilon \in G, \quad (x, y) \in \mathbb{C}^2$$

It has a fixed point  $p_1 = (0, 0)$  at  $\mathbb{C}^2$  (and two other fixed points  $p_2 = [0 : 1 : 0], p_3 = [1 : 0 : 0]$  at infinity). For each  $\epsilon \in G$  we have  $\epsilon^*(\omega) = \epsilon^{d+1}\omega$  and so  $G$  leaves  $\mathcal{F}_d$  invariant. We have

$$\text{Sing}(\mathcal{F}_d)_{\mathbb{C}} = \{(\epsilon, \epsilon^{-d}) \mid \epsilon \in G\}$$

(there is no singularity at infinity) and  $G$  acts on  $\text{Sing}(\mathcal{F}_d)$  transitively.

## 2.2 Compactification

In differential and algebraic geometry many theorems are stated for compact varieties and so it would be of interest to compactify the ambient space of our foliations.

The projective space of dimension  $n$  is defined as follows:

$$\mathbb{P}^n = (\mathbb{C}^{n+1} - \{0\}) / \sim$$

where

$$a, b \in \mathbb{C}^{n+1} - \{0\}, a \sim b \Leftrightarrow a = kb, \text{ for some } k \in \mathbb{C} - \{0\}.$$

It turns out that  $\mathbb{P}^n$  is a complex manifold. We will mainly use  $\mathbb{P}^1$  and  $\mathbb{P}^2$ . The projective space of dimension one  $\mathbb{P}^1$  is covered by two charts  $x, x'$  biholomorphic to  $\mathbb{C}$  and the transition map is given by

$$x' = \frac{1}{x}.$$

The projective space of dimension two  $\mathbb{P}^2$  is covered by three charts  $(x, y), (u, v), (u', v')$  biholomorphic to  $\mathbb{C}^2$  and the transition maps are given by

$$v = \frac{y}{x}, u = \frac{1}{x}, \quad v' = \frac{x}{y}, u' = \frac{1}{y}.$$

Considering the chart  $(\mathbb{C}^2, (x, y))$ ,  $\mathbb{P}^2$  becomes a compactification of  $\mathbb{C}^2$ . A foliation  $\mathcal{F}(\omega)$ ,  $\omega = Pdy - Qdx$  extends to a holomorphic foliation in  $\mathbb{P}^2$  in a unique way. For instance, in the chart  $(u, v)$  we have

$$\omega = P\left(\frac{1}{u}, \frac{v}{u}\right)d\left(\frac{v}{u}\right) - Q\left(\frac{1}{u}, \frac{v}{u}\right)d\left(\frac{1}{u}\right) = \frac{\tilde{P}(u, v)dv - \tilde{Q}(u, v)du}{u^{d+2}}, \quad \tilde{P}, \tilde{Q} \in \mathbb{C}[x, y].$$

Another compactification of  $\mathbb{C}^2 = \mathbb{C} \times \mathbb{C}$  is  $\mathbb{P}^1 \times \mathbb{P}^1$  which is useful for studying the Ricatti foliations given by:

$$\omega = q(x)dy - (p_0(x) + p_1(x)y + p_2(x)y^2)dx, \quad p_0, p_1, p_2, q \in \mathbb{C}[x].$$

Substituting  $y = \frac{1}{y'}$  we have

$$\omega = \frac{1}{y'^2}(-q(x)dy - (p_0(x)y^2 + p_1(x)y + p_2(x))dx)$$

and so all the projective lines  $\{a \in \mathbb{C} \mid q(a) \neq 0\} \times \mathbb{P}^1$  are transversal to the foliation. This will be later used to define the global holonomy of Ricatti foliations.

One usually uses the projective degree for a holomorphic foliation in  $\mathbb{P}^2$ . It is the number of tangency points of a generic line with the foliation. In this way we may redefine  $\mathcal{F}(d)$  to be the set of holomorphic foliations of projective degree  $d$  in  $\mathbb{P}^2$ .

## 2.3 Holonomy

A leaf  $L$  of a holomorphic foliation  $\mathcal{F}$  in  $\mathbb{P}^2$  is a Riemann surface which in principle may have infinite genus, i.e. the homotopy group  $\pi_1(L, p)$ ,  $p \in L$  may not be finitely generated. We take a transversal to  $\mathcal{F}$  section  $\Sigma$  at  $p$ . This is the image of an embedding of  $(\mathbb{C}, 0)$  to  $\mathbb{P}^2$  such that it maps 0 to  $p$  and is transverse to the leaves of  $\mathcal{F}$ . For instance one can take  $\Sigma$  a piece of line which crosses  $p$  and is not tangent to  $L$ . For  $\gamma \in \pi_1(L, p)$  we can define the holonomy

$$h_\gamma : \Sigma \rightarrow \Sigma$$

of  $\mathcal{F}$  along  $\gamma$  in the same way that we have done it for the Poincaré first return map.

## 2.4 Algebraic solutions and algebraic limit cycles

For a reduced polynomial  $f \in \mathbb{C}[x, y]$  we consider the situation in which the curve

$$L_f = L_{f, \mathbb{C}} = \{f = 0\} := \{(x, y) \in \mathbb{C}^2 \mid f(x, y) = 0\}$$

is invariant under a foliation  $\mathcal{F}(\omega)$ .

**Example 2.2.** For various values of  $t \in \mathbb{R}$  the set

$$L_{f-t, \mathbb{R}}, f =: \frac{y^2}{2} + \frac{(x^2 - 1)^2}{4}$$

is depicted in 3.1. All of these curves are invariant by  $\mathcal{F}(df)$ .

Let us find an algebraic statement for the mentioned fact. For a smooth point  $a \in L_f$ ,  $\omega$  restricted to the tangent space of  $L_f$  at  $a$  is zero. Since  $df$  has also the same property, we conclude that  $\omega \wedge df$  restricted to the curve  $L_f$  is zero. But  $\omega \wedge df = Pdx \wedge dy$  for some polynomial  $P \in \mathbb{C}[x, y]$  and so  $P$  restricted to  $L_f$  is zero. Since  $f$  is reduced we conclude that

$$\omega \wedge df = f\theta, \theta \in \Omega_{\mathbb{C}^2}^2.$$

Conversely, if the above equality holds the curve  $L_f$  is invariant under  $\mathcal{F}(\omega)$ . We call  $L_f$  an algebraic leaf of  $\mathcal{F}(\omega)$ .

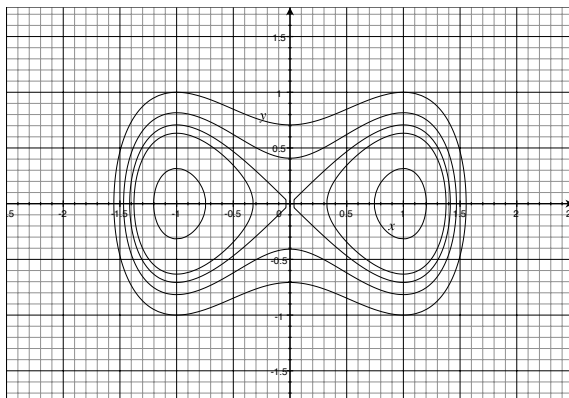


Figure 2.1: Eight:  $\frac{y^2}{2} + \frac{(x^2-1)^2}{4} = t$ ,  $t = \frac{1}{20}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}$

**Definition 2.1.** A holomorphic foliation  $\mathcal{F}(\omega)$  has a (meromorphic) first integral if there is a rational function  $\frac{F}{G}$ ,  $F, G \in \mathbb{C}[x, y]$  such that  $\mathcal{F} = \mathcal{F}(GdF - FdG)$ .  $\frac{F}{G}$  is called the first integral of  $\mathcal{F}$ . All the algebraic curves  $\frac{F}{G} = t, t \in \mathbb{C}$  are invariant by  $\mathcal{F}$ .

**Theorem 2.2.** (Darboux) A holomorphic foliation  $\mathcal{F}(\omega)$  has a finite number of algebraic leaves.

Now let us come back to the real context,  $f \in \mathbb{R}[x, y]$ ,  $\omega \in \Omega_{\mathbb{R}^2}^1$ . The real curve  $L_{f, \mathbb{R}}$  may have many connected components and in particular it may have many ovals. Take one of them and call it  $\delta$ . If  $\mathcal{F}(\omega)$  leaves  $L_f$  invariant then  $\delta$  is a leaf of  $\mathcal{F}_{\mathbb{R}}$  and so it has a holonomy:

**Definition 2.2.** If the holonomy of  $\delta$  is not the identity map then it is called an algebraic limit cycle.

By Darboux theorem it follows that a foliation  $\mathcal{F}(\omega)_{\mathbb{R}}$  has a finite number of algebraic limit cycles. However, the second part of the Hilbert 16-th problem for algebraic limit cycles is not yet solved.

**Problem 2.1.** (Hilbert 16-th problem for algebraic limit cycles) Fix a natural number  $n \in \mathbb{N}$ . Is there some natural number  $N(n) \in \mathbb{N}$  such that each foliation  $\mathcal{F}(Pdx - Qdy)_{\mathbb{R}}$ ,  $P, Q \in \mathbb{R}[x, y]$  with  $\deg(P), \deg(Q) \leq n$  has at most  $N(n)$  algebraic limit cycles.

We note that the above problem even for the case  $n = 2$  is open.

**Example 2.3.** The foliation

$$\mathcal{F}(nxdy + mydx), \quad n, m \in \mathbb{N}$$

has the first integral  $x^m y^n$ . This example shows that the degree of a foliation can be small but the degree of its algebraic solutions can be big.

## 2.5 Lins-Neto's examples

It is a classical fact in real algebraic geometry that a real algebraic curve of genus  $g$  has at most  $g+1$  ovals in the real plane  $\mathbb{R}^2$ . In order to prove the algebraic Hilbert 16-th problem we may try to prove that the genus of algebraic leaves of holomorphic foliations of degree is uniformly bounded. Unfortunately, this affirmation is wrong. The first examples may be derived from

**Exercise 2.1.** There is a Hurwitz-Zeuthen genus formula for calculating the genus of singular algebraic curves. Apply this formula for

$$x^n y^m (x + y - 1)^k = t, \quad n, m, k \in \mathbb{N}, \quad t \in \mathbb{C}$$

and conclude that its genus can go to the infinity.

The pencils  $\mathcal{F}_{i,t} = \mathcal{F}(\omega_i + t\eta_i)$ ,  $i = 1, 2$ ,  $t \in \mathbb{P}^1$ , where

$$\omega_1 = (4x - 9x^2 + y^2)dy - 6y(1 - 2x)dx, \quad \eta_1 = 2y(1 - 2x)dy - 3(x^2 - y^2)dx,$$

$$\omega_2 = y(x^2 - y^2)dy - 2x(y^2 - 1)dx, \quad \eta_2 = (4x - x^3 - x^2y - 3xy^2 + y^3)dy + 2(x + y)(y^2 - 1)dx$$

are studied by A. Lins Neto in [10]. They satisfy

$$d\omega_i = \alpha_i \wedge \omega_i, \quad i = 1, 2,$$

where

$$\alpha_i := \lambda_i \frac{dQ_i}{Q_i}, \quad \lambda_1 = \frac{5}{6}, \quad \lambda_2 = \frac{3}{4},$$

$$Q_1 = -4y^2 + 4x^3 + 12xy^2 - 9x^4 - 6x^2y^2 - y^4, \quad Q_2 = (y^2 - 1)(x + 2 + y^2 - 2x)(x^2 + y^2 + 2x).$$

**Theorem 2.3.** (*Lins Neto*) *The set*

$$E_i = \{t \in \mathbb{P}^1 \mid \mathcal{F}_{i,t} \text{ has a meromorphic first integral}\}$$

is  $\mathbb{Q} + \mathbb{Q}e^{2\pi i/3}$  for  $i = 1$  and is  $\mathbb{Q} + i\mathbb{Q}$  for  $i = 2$ . Moreover, for any natural number  $n$  the set of points  $t$  in  $E_i$  such that the algebraic solutions of  $\mathcal{F}(\omega_i + t\eta_i)$  are of degree (resp. genus) less than  $n$ , is finite.

## 2.6 Minimal set

For a holomorphic foliation in  $\mathbb{P}^2$  one may formulate many problems related to the accumulation of its leaves. The most simple one which is still open is the following:

**Problem 2.2.** Is there a foliation  $\mathcal{F}$  in  $\mathbb{P}^2$  with a leaf  $L$  which does not accumulate in the singularities of  $\mathcal{F}$ .

For instance the above problem for Jouanolou foliation is proved numerically for  $d \leq 4$  and it is still open for general  $d$ .

Let us suppose that such an  $\mathcal{F}$  and  $L$  exist and set  $M := \bar{L}$ , where the closure is taken in  $\mathbb{P}^2$ . It follows that  $M$  is a union of leaves of  $\mathcal{F}$ . We may suppose that  $M$  does not contain a proper  $\mathcal{F}$ -invariant subset. In this case we call  $M$  a minimal set.

**Proposition 2.1.** *A foliation in  $\mathbb{P}^2$  with algebraic leaf has not a minimal set.*

For many other useful statements on minimal sets see [1]. For local theory of holomorphic foliations see [2].

## Chapter 3

# Abelian integrals

The objective of this section is to introduce abelian integrals in the context of holomorphic foliations. The main problem in this direction is Arnold-Hilbert infinitesimal problem.

### 3.1 Continuous family of cycles

Let us be given a polynomial  $f \in \mathbb{R}[x, y]$  of degree  $d$ . We consider  $f$  as a function from  $\mathbb{C}^2$  to  $\mathbb{C}$ . The set of critical points and critical values of  $f$  are defined to be

$$P_{\mathbb{R}} = \{(a, b) \in \mathbb{R}^2 \mid f_x(a, b) = f_y(a, b) = 0\}, \quad C_{\mathbb{R}} := f(P_{\mathbb{R}}).$$

Let us take two consecutive elements of  $C$ , namely  $c_1 < c_2$ , and assume that the real affine variety

$$L_{t, \mathbb{R}} := f^{-1}(t), \quad c_1 \leq t \leq c_2$$

contains a connected component  $\delta_t$ , namely an oval, such that it is a closed cycle in  $\mathbb{R}^2$ , varies continuously with respect to the parameter  $t$  and for  $t = c_1, c_2$  it is either a point or a closed polygon of paths.

**Example 3.1.** To carry an example in mind, take the polynomial  $f = y^2 - x^3 + 3x$  in two variables  $x$  and  $y$ . We have  $C = \{-2, 2\}$ . For  $t$  a real number between 2 and  $-2$  the level surface of  $f$  in the real plane  $\mathbb{R}^2$  has two connected pieces which one of them is an oval and we can take it as  $\delta_t$ . In this example as  $t$  moves from  $-2$  to 2,  $\delta_t$  is born from the critical point  $(-1, 0)$  of  $f$  and end up in the  $\alpha$ -shaped piece of the fiber  $f^{-1}(2) \cap \mathbb{R}^2$  (See Figure 3.1).

**Example 3.2.** Another good example is

$$f = \frac{y^2}{2} + \frac{(x^2 - 1)^2}{4}$$

The set of critical values of  $f$  is  $C = \{0, \frac{1}{4}\}$  and we can distinguish three family of ovals.

### 3.2 Abelian integrals

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a polynomial mapping and  $\delta_t \cong \mathbb{S}^1$ ,  $t \in (\mathbb{R}, 0)$  be a continuous family of ovals in the fibers of  $f$ . The level surfaces of  $f$  are the images of the solutions of the

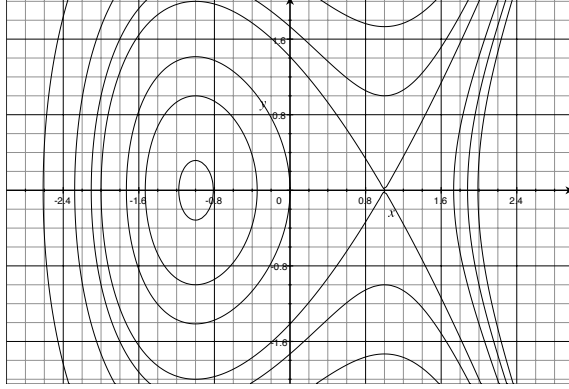


Figure 3.1: Elliptic curves:  $y^2 - x^3 + 3x - t$ ,  $t = -1.9, -1, 0, 2, 3, 5, 10$

Hamiltonian ordinary differential equation

$$(3.1) \quad \mathcal{F}_0 : \begin{cases} \dot{x} = f_y \\ \dot{y} = -f_x \end{cases} .$$

We make a perturbation of  $\mathcal{F}_0$

$$(3.2) \quad \mathcal{F}_\epsilon : \begin{cases} \dot{x} = f_y + \epsilon P(x, y) \\ \dot{y} = -f_x + \epsilon Q(x, y) \end{cases} , \epsilon \in (\mathbb{R}, 0),$$

where  $P$  and  $Q$  are two polynomials with real coefficients.

Usually one expects that in the new ordinary differential equation the cycle  $\delta_0$  breaks and accumulates, in positive or negative time, in some part of the real plane or infinity. However, we have the following proposition:

**Proposition 3.1.** *If the Abelian integral*

$$I(t) := \int_{\delta_t} (Pdy - Qdx)$$

*is zero for  $t = 0$  but not identically zero then for any small  $\epsilon$  there will be a limit cycle of  $\mathcal{F}_\epsilon$  near enough to  $\delta_0$ .*

The proof is classical and can be found in [8, 14, 13]. If the Abelian integral is identically zero (for instance if  $\delta_t$  is homotopic to zero in the complex fiber of  $f$ ) then the birth of limit cycles is controlled by iterated integrals (see for instance [6, 15]). In our main example take the ordinary differential equation

$$(3.3) \quad \mathcal{F}_\epsilon : \begin{cases} \dot{x} = 2y + \epsilon \frac{x^2}{2} \\ \dot{y} = 3x^2 - 3 + \epsilon sy \end{cases} , \epsilon \in (\mathbb{R}, 0).$$

If  $\int_{\delta_0} (\frac{x^2}{2} dy - sy dx) = 0$  or equivalently

$$s := \frac{-\int_{\Delta_0} x dx \wedge dy}{\int_{\Delta_0} dx \wedge dy} = \frac{\Gamma(\frac{5}{12})\Gamma(\frac{13}{12})}{\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})} \sim 1.2636$$

where  $\Delta_0$  is the bounded open set in  $\mathbb{R}^2$  with the boundary  $\delta_0$ , then for  $\epsilon$  near to 0,  $\mathcal{F}_\epsilon$  have a limit cycle near  $\delta_0$ . In fact for  $\epsilon = 1$  and  $s = 0.9$  such a limit cycle still exists and it is depicted in Figure (1.1).

### 3.3 Arnold-Hilbert Problem

A weaker version of H16, known as the infinitesimal Arnold-Hilbert problem asks for a reasonable bound for the number of zeros of real Abelian integrals when the degrees of  $f$  and  $P, Q$  are bounded.

**Problem 3.1.** (Infinitesimal Arnold-Hilbert problem) Determine the number

$$Z(m, n) := \max\#\left\{\int_{\delta_t} \omega = 0 \mid f \in \mathbb{R}[x, y] \ \omega \in \Omega_{\mathbb{R}^2}^1, \deg(f) \leq m, \deg(\omega) \leq n\right\},$$

where  $\delta_t$  is a continuous family of ovals in the real level surfaces of  $f$ .

There are some partial solutions to this problem but it is still open in its generality (see [9, 5]). Using the theory of fewnomials Khovanski has proved that  $Z(m, n)$  is a finite number. It is proved by Ilyasheno, Yakovenko and others that  $Z(m, n)$  has an exponential growth.

**Theorem 3.1.** (*Gavrilov-Horozov-Iliev, [5]*)

$$N(3, 2) = 2.$$

### 3.4 Arnold-Hilbert problem in dimension zero

Even the zero dimensional version of Arnold-Hilbert problem, in which Abelian integrals are algebraic functions, is not completely solved(see [7]).

Let us be given a degree  $d$  polynomial  $f(x) \in \mathbb{R}[x]$  in one variable  $x$ . Let  $c_1$  and  $c_2$  be two real critical values of  $f$  such that between  $c_1$  and  $c_2$  there is no more critical value of  $f$  and  $f^{-1}(t)$  for  $t$  between  $c_1$  and  $c_2$  contains two real points  $x_1$  and  $x_2$  such that for  $t$  near  $c_1$  they collapse into each other in a critical point  $p_1$  with  $f(p_1) = c_1$ . In another word we have a zero dimensional cycle which vanishes at  $p_1$ .

**Definition 3.1.** The zero dimensional abelian integral is defined to be

$$\int_{\delta_t} \omega := \omega(x_2) - \omega(x_1), \ \delta_t := [x_2] - [x_1], \ \omega \in \mathbb{R}[x].$$

We define  $N_0(m, n)$  in a similar way as we have defined in the previous section.

**Proposition 3.2.** *We have*

$$(3.4) \quad n - 1 - \left\lfloor \frac{n}{m} \right\rfloor \leq Z(m, n) \leq \frac{(m-1)(n-1)}{2}.$$

The lower bound in this inequality is given by the dimension of the vector space of Abelian integrals

$$V_n = \left\{ \int_{\delta_t} \omega, \ \deg \omega \leq n \right\}$$

where  $f$  is a fixed general polynomial of degree  $m$ , while the upper bound is a reformulation of Bezout's theorem. When  $m = 3$  we get  $Z(m, m-1) = 1$ . The space of Abelian integrals  $V_n$  is Chebishev, possibly with some accuracy.

**Definition 3.2.** Recall that  $V_n$  is said to be Chebishev with accuracy  $c$  if every  $I \in V_n$  has at most  $\dim V_n - 1 + c$  zeros in the domain  $D$ .

**Problem 3.2.** Give an exact description of the number  $N_0(d, d - 1)$ .

If we do not put any restriction on  $x_1$  and  $x_2$  the sharp upper bound is  $(d-1)(d-2)/2$ . For instance take  $f(x) = (x-1)(x-2)\dots(x-d)$  and  $g = (x-1)(x-2)\dots(x-d+1)$ . In the image of  $f$  we can find  $d-1$  intervals with mentioned property and the main problem is: How many of those zeros can be grouped in one of such intervals.

# Chapter 4

## Center problem

In this chapter we consider the space of holomorphic foliation in  $\mathbb{P}^2$  which have at least one center singularity (consequently we will use the projective degree). It turns out that such a space is algebraic. The classification of all irreducible components of such an algebraic variety is known as the center (or center-focus) problem in the literature. The main source for the content of this chapter is [13]

### 4.1 Center singularity

Let  $\mathcal{F}$  be a germ of singular foliation at  $(\mathbb{C}^2, 0)$ . We say that  $0 \in \mathbb{C}^2$  is a center singularity of  $\mathcal{F}$  or simply a center of  $\mathcal{F}$ , if there exists a germ of holomorphic function  $f : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}, 0)$  which has non-degenerate critical point at  $0 \in \mathbb{C}^2$ , and the leaves of  $\mathcal{F}$  near 0 are given by  $f = \text{const.}$ . The point 0 is also called a Morse singularity of  $f$ . Morse lemma in the complex case implies that there exists a local coordinate system  $(x, y)$  in  $(\mathbb{C}^2, 0)$  with  $x(0) = 0$ ,  $y(0) = 0$  and such that  $f(x, y) = x^2 + y^2$ . Near the center the leaves of  $\mathcal{F}$  are homeomorphic to a cylinder, therefore each leaf has a nontrivial closed cycle that will be called the Lefschetz vanishing cycle.

### 4.2 Center variety

Let  $\mathcal{F}(d)$  be the space of degree  $d$  holomorphic foliations in  $\mathbb{P}^2$  and  $\mathcal{M}(d)$  the closure of the set of foliations of degree  $d$  and with at least one center in  $\mathcal{F}(d)$ . The following example gives us a huge number of these foliations:

**Example 4.1.** Let  $\tau : \mathbb{P}^2 \rightarrow M$  be a holomorphic map between  $\mathbb{P}^2$  and a complex compact manifold  $M$  with  $\dim(M) \geq 3$ . Let also  $\mathcal{F}$  be a holomorphic codimension one singular foliation in  $M$ . We say that  $\tau$  has a tangency point  $a \in \mathbb{P}^2$  of order two with  $\mathcal{F}$  if  $\tau(a)$  is a regular point of the foliation  $\mathcal{F}$ ,  $\mathcal{F}$  in a coordinate  $(x, y) \in (\mathbb{C}^{n-1}, 0) \times (\mathbb{C}, 0)$  around  $a$  is given by  $y = \text{const.}$  and  $a$  is a non-degenerate critical point of  $y \circ \tau$ . This says that the pullback foliation  $\tau^*(\mathcal{F})$  has a center at the point  $a$ .

I have learned the statement and proof of the following proposition from A. Lins Neto.

**Proposition 4.1.**  $\mathcal{M}(d)$  is an algebraic subset of  $\mathcal{F}(d)$ .

*Proof.* Let  $\mathcal{M}_0(d)$  be the set of all foliations in  $\mathcal{M}(d)$  with a center at the origin  $(0,0) \in \mathbb{C}^2 \subset \mathbb{P}^2$  and with a local first integral of the type

$$(4.1) \quad f = xy + f_3 + f_4 + \cdots + f_n + h.o.t.$$

in a neighborhood of  $(0,0)$ . Let us prove that  $\mathcal{M}_0(d)$  is an algebraic subset of  $\mathcal{F}(d)$ .

Let  $\mathcal{F}(\omega) \in \mathcal{M}_0(d)$  and  $\omega = \omega_1 + \omega_2 + \omega_3 + \cdots + \omega_{d+1}$  be the homogeneous decomposition of  $\omega$ , then in a neighborhood around  $(0,0)$  in  $\mathbb{C}^2$ , we have

$$\omega \wedge df = 0 \Rightarrow (\omega_1 + \omega_2 + \omega_3 + \cdots + \omega_{d+1}) \wedge (d(xy) + df_3 + df_4 + \cdots) = 0$$

Putting the homogeneous parts of the above equation equal to zero, we obtain

$$(4.2) \quad \begin{cases} \omega_1 \wedge d(xy) = 0 \Rightarrow \omega_1 = k.d(xy), \text{ } k \text{ is constant} \\ \omega_1 \wedge df_3 = -\omega_2 \wedge d(xy) \\ \dots \\ \omega_1 \wedge df_n = -\omega_2 \wedge df_{n-1} - \cdots - \omega_{n-1} \wedge d(xy) \\ \dots \end{cases}$$

Dividing the 1-form  $\omega$  by  $k$ , we can assume that  $k = 1$ . Let  $\mathcal{P}_n$  denote the set of homogeneous polynomials of degree  $n$ . Define the operator :

$$S_n : \mathcal{P}_n \rightarrow (\mathcal{P}_n dx \wedge dy)$$

$$S_n(g) = \omega_1 \wedge d(g)$$

We have

$$\begin{aligned} S_{i+j}(x^i y^j) &= d(xy) \wedge d(x^i y^j) = (xdy + ydx) \wedge (x^{i-1} y^j + j x^i y^{j-1}) \\ &= (j - i) x^i y^j dx \wedge dy \end{aligned}$$

This implies that when  $n$  is odd  $S_n$  is bijective and so in (4.2),  $f_n$  is uniquely defined by the terms  $f_m, \omega_m$ 's  $m < n$ , and when  $n$  is even

$$Im(S_n) = A_n dx \wedge dy$$

where  $A_n$  is the subspace generated by the monomials  $x^i y^j$ ,  $i \neq j$ . When  $n$  is even the existence of  $f_n$  implies that the coefficient of  $(xy)^{\frac{n}{2}}$  in

$$-\omega_2 \wedge df_{n-1} - \cdots - \omega_{n-1} \wedge d(xy)$$

which is a polynomial, say  $P_n$ , with variables

$$\text{coefficients of } \omega_2 \dots \omega_{n-1}, f_2, \dots, f_{n-1}$$

is zero. The coefficients of  $f_i, i \leq n-1$  is recursively given as polynomials in coefficients of  $\omega_i, i \leq n-1$  and so the algebraic set

$$X : P_4 = 0 \ \& \ P_6 = 0 \ \& \ \dots \ \& \ P_n = 0 \ \dots$$

consists of all foliations  $\mathcal{F}$  in  $\mathcal{F}(d)$  which have a formal first integral of the type 4.1 at  $(0,0)$ . From a result of Mattei and Moussu (theorem A, [11]), it follows that  $\mathcal{F}$  has a holomorphic first integral of the type (4.1). This implies that  $\mathcal{M}_0(d) = X$  is algebraic. Note that by Hilbert zeroes theorem, a finite number of  $P_i$ 's defines  $\mathcal{M}_0(d)$ . The set  $\mathcal{M}(d)$  is obtained by the action of the group of automorphisms of  $\mathbb{P}^2$  on  $\mathcal{M}_0(d)$ . Since this group is compact we conclude that  $\mathcal{M}(d)$  is also algebraic.  $\square$

### 4.3 Components of holomorphic foliations with a first integral

Let  $\mathcal{P}_d$  be the set of polynomials of maximum degree  $d$  in  $\mathbb{C}^2$  and

$$(F, G) \in \mathcal{P}_{a+1} \times \mathcal{P}_{b+1}, \quad \frac{a+1}{b+1} = \frac{q}{p}, \quad g.c.d.(p, q) = 1, \quad \mathcal{R} := \{F = 0, G = 0\} \subset \mathbb{P}^2$$

The foliation  $\mathcal{F} = \mathcal{F}(pGdF - qFdG)$  has the first integral:

$$f : \mathbb{P}^2 \setminus \mathcal{R} \rightarrow \mathcal{S}, \quad f(x, y) = \frac{F(x, y)^p}{G(x, y)^q}$$

i.e., the leaves of the foliation  $\mathcal{F}$  are contained in the level surfaces of  $f$ . Let  $\mathcal{I}(a, b)$  be the closure of the set of the mentioned holomorphic foliations in  $\mathcal{F}(d)$ .

**Theorem 4.1.** ([13, 12])  $\mathcal{I}(a, b), a + b > 2$  is an irreducible component of  $\mathcal{M}(d)$ , where  $d = a + b$ .

We can restate our main theorem as follows: Let  $\mathcal{F} \in \mathcal{I}(a, b)$ ,  $p$  one of the center singularities of  $\mathcal{F}$  and  $\mathcal{F}_t$  a holomorphic deformation of  $\mathcal{F}$  in  $\mathcal{F}(d)$ , where  $d = a + b$ , such that its unique singularity  $p_t$  near  $p$  is still a center.

**Theorem 4.2.** In the above situation, if  $a + b > 2$  then there exists an open dense subset  $U$  of  $\mathcal{I}(a, b)$ , such that for all  $\mathcal{F}(pGdF - qFdG) \in U$ ,  $\mathcal{F}_t$  admits a meromorphic first integral. More precisely, there exist polynomials  $F_t$  and  $G_t$  such that  $\mathcal{F}_t = \mathcal{F}(pG_t dF_t - qF_t dG_t)$ , where  $F_t$  and  $G_t$  are holomorphic in  $t$  and  $F_0 = F$  and  $G_0 = G$ .

This theorem also says that the persistence of one center implies the persistence of all other centers and dicritical singularities (the points of  $\{F = 0\} \cap \{G = 0\}$ ).

Let  $\mathcal{F}$  be a foliation in  $\mathbb{P}^2$  given by the polynomial 1-form

$$(4.3) \quad \omega(f, \lambda) = \omega(f_1, \dots, f_r, \lambda_1, \dots, \lambda_r) = f_1 \cdots f_r \sum_{i=1}^r \lambda_i \frac{df_i}{f_i}$$

where the  $f_i$ 's are irreducible polynomials in  $\mathbb{C}^2$ ,  $\sum d_i \lambda_i = 0$  and  $d_i = \deg(f_i)$ .  $\mathcal{F}$  is called a logarithmic foliation and it has the multi-valued first integral  $f = f_1^{\lambda_1} \cdots f_r^{\lambda_r}$  in  $U = \mathbb{P}^2 \setminus (\cup_{i=1}^r \{f_i = 0\})$ . We can prove that generically, the degree of  $\mathcal{F}$  is  $d = \sum_{i=1}^r d_i - 2$  and has  $d^2 + d + 1 - \sum_{1 \leq i < j \leq r} d_i d_j$  centers.

Let  $\mathcal{L}(d_1, d_2, \dots, d_r)$  be the set of all logarithmic foliations of the above type.

**Theorem 4.3.** ([14]) If at least one of  $d_i$  is one then  $\mathcal{L}(d_1, d_2, \dots, d_r)$  is an irreducible component of  $\mathcal{M}(d)$ , where  $d = \sum_{i=1}^r d_i - 2$ .

The classification of degree two polynomial differential equations was done by Dulac in [4]. Going to the language of holomorphic foliations in  $\mathbb{P}^2$ , instead of using the language of polynomial differential equations, this classification was completed in [3] for degree two holomorphic foliation in  $\mathbb{P}^2$ . Deformation of real Hamiltonian equations with a center singularity, generating limit cycles from the Lefschetz vanishing cycles of the center, has been one of the methods of approach to Hilbert sixteen problem, for this see Roussarie's book [16] and its reference. Yu.S. Ilyashenko in [8] shows that the persistence of a center after deformation of a generic Hamiltonian equation implies that the deformed equation is also Hamiltonian. He uses this fact to get a certain number of limit cycles after deformation of Hamiltonian equations.

## 4.4 The center variety and Limit cycles

Let  $X$  be an irreducible component of  $\mathcal{M}(d)$ ,  $\mathcal{F} \in X - \text{sing}(\mathcal{M}(d))$  be a real foliation, i.e. its equation has real coefficients,  $p$  be a real center singularity and  $\delta_t, t \in (\mathbb{R}, 0)$  be a family of real vanishing cycles around  $p$ . The cyclicity of  $\delta_0$  in a deformation of  $\mathcal{F}$  inside  $\mathcal{F}(d)$  is greater than  $\text{codim}_{\mathcal{F}(d)}(X) - 1$ . Roughly speaking, the cyclicity of  $\delta_0$  is the maximum number of limit cycles appearing near  $\delta_0$  after a deformation of  $\mathcal{F}$  in  $\mathcal{F}(d)$ . The proof of this fact and the exact definition of cyclicity can be found in [16]. We have  $\text{codim}_{\mathcal{F}(d)}(\mathcal{L}(d+1)) - 1 = \frac{(d+2)(d+1)}{2}$  and this is the number obtained by Yu. Ilyashenko in [8]. One can state similar results for the components obtained in Theorem 4.1 and 4.3.

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